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Kinetic Theory for a Dilute Gas of Particles with Spin

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The kinetic theory of particles with spin previously developed for a LORENTZian gas is extended to the case of a pure gas. In part A the transport (BOLTZMANN) equation for the one particle distribution operator is stated and discussed (conservation laws, H-theorem). A magnetic field acting on the magnetic moment of the particles is incorporated throughout. In part B the pertaining linearized collision operator and certain bracket expressions linked with this operator are considered. Part C deals with the expansion of the distribution operator and of the linearized transport equation with respect to a complete set of composite irreducible tensors built from the components of particle velocity and spin. Thus, the distribution operator is replaced by a set of tensors depending only on time and space-coordinates. The physical meaning of these tensors (expansion coefficients) is invoked. They obey a set of coupled first-order differential equations (transport-relaxation equations). The reciprocity relations for the relaxation matrices are stated. Finally a detailed discussion of angular momentum conservation is given.

The transport equation for particles with spin, first derived by one of us[1] and again by SNIDER[2], differs from the classical BOLTZMANN equation for the monatomic gas in two respects: (1) the classical one particle distribution function is replaced by a function dependent on the spin operator too or — what is the same — by a matrix with respect to spin indices; (2) the collision term contains the scattering amplitude and its adjoint in a bilinear way rather than simply a scattering cross section.

As the simplest application the diffusion of spin one half particles through an irregular lattice (LORENTZian gas) has been studied[3]. The structure of the distribution matrix was investigated in detail; neglecting its off-diagonal elements might cause an error in the diffusion coefficient comparable with the error caused by neglecting the effect of non-spherical (spin-dependent) interaction on the cross section itself[4, 5]. In addition to this, the diffusion of spin one half particles in the precence of a homogeneous magnetic field was studied by WALDMANN and KUPATT[6].

The generalization of these methods to a pure gas has been overdue for long and will now be given in this paper and a subsequent one. The purpose is to find the relations between the transport coefficients and the various functions of scattering angle and energy which occur in the binary scattering amplitude. The relations to be derived replace and amplify the well-known CHAPMAN–ENSKOG Ω-integrals for monatomic gases without spin. The procedure followed is in principle the same as with the LORENTZian gas[3], but in the case of arbitrary spin it is somewhat more laborious to write down the details. And of course the physical contents are richer. At the same time it is nice to see how naturally the methods of classical kinetic theory are extended to this quantum case.

The pure gas of particles with spin has meanwhile also been treated by MCCOURT and SNIDER[7] according to a method which is closely akin to the original CHAPMAN–ENSKOG theory. We think that the relaxation formalism used in the present paper is desirable too and gives good insight into the physics involved. Furthermore, we have tried to give, without restrictive assumptions, a general treatment of the somewhat subtle theoretical questions arising in connection with angular momentum conservation.

Of course, the case of particles with spin has its classical counterpart. The classically rotating molecules are even more general in so far as the magnitude of their intrinsic molecular angular momentum is not

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[1] L. WALDMANN, Z. Naturforschg. **12 a**, 660 [1957]; **13 a**, 609 [1958].

[2] R. F. SNIDER, J. Chem. Phys. **32**, 1051 [1960].

[3] L. WALDMANN, Nuovo Cim. **14**, 898 [1959]; Z. Naturforschg. **15 a**, 19 [1960].

[4] L. WALDMANN, in Proceedings of the International Seminar on the Transport Properties of Gases, Brown University, Providence R. I. [1964], p. 59.

[5] L. WALDMANN, Quantum-Theoretical Transport Equations for Polyatomic Gases, in Statistical Mechanics of Equilibrium and Non-Equilibrium, ed. J. MEIXNER, Amsterdam 1965, p. 177.

[6] L. WALDMANN and H.-D. KUPATT, Z. Naturforschg. **18 a**, 86 [1963].

[7] F. R. MC COURT and R. F. SNIDER, J. Chem. Phys. **41**, 3185 [1964].

fixed. Thus it is useful to compare the generalities of the present paper also with those of the classical models, e. g. with WALDMANN's work[8] on the LORENTZian gas of rotating molecules and with the paper by CONDIFF, LU, and DAHLER[9] on the pure gas of rough spheres.

A) Transport Equation for a Gas Consisting of Particles with Spin

§ 1. Basic notations. Transport equation

We describe a particle in the gas by its position vector $\boldsymbol{x}$, its velocity $\boldsymbol{v}$ and its spin vector $\hbar\,\boldsymbol{s}$. Position and velocity are treated as classical variables, the spin is treated as a quantum mechanical operator. The three components of $\boldsymbol{s}$ can be represented in the usual way by $(2\,S+1)\times(2\,S+1)$ matrices where $\hbar\,S$ is the magnitude of the spin (intrinsic angular momentum). The Cartesian components of $\boldsymbol{s}$ obey the fundamental commutation relations[10] $(\mu,\nu,\lambda=1,2,3)$

$$s_\mu\,s_\nu - s_\nu\,s_\mu = i\,\varepsilon_{\mu\nu\lambda}\,s_\lambda \tag{1.1}$$

and
$$\boldsymbol{s}\cdot\boldsymbol{s} = S(S+1)\,. \tag{1.2}$$

The macroscopic state of the gas is described by the Hermitian one-particle distribution matrix with elements

$$f_{MM'}(t,\boldsymbol{x},\boldsymbol{v})$$

where M, M' are spin indices, running from $-S$ to $+S$. It is perhaps more appealing to write the distribution matrix in operator form indicating the dependence on $\boldsymbol{s}$:

$$f(t,\boldsymbol{x},\boldsymbol{v},\boldsymbol{s})\,. \tag{1.3}$$

This distribution operator is a polynomial in the three components of the spin vector $\boldsymbol{s}$. Its coefficients depend on $t,\boldsymbol{x},\boldsymbol{v}$ and are real. No powers in s_μ higher than $2\,S$ will occur due to (1.1) and (1.2).

The physical meaning of the distribution operator (1.3) may be inferred from the v. NEUMANN expectation value formula. Especially the local particle density n is given by

$$n(t,\boldsymbol{x}) = \operatorname{tr}\int f(t,\boldsymbol{x},\boldsymbol{v},\boldsymbol{s})\;\mathrm{d}^3v \tag{1.4}$$

where "tr" means the trace to be taken on spin matrices. The local mean value of any operator $\Psi(\boldsymbol{x},\boldsymbol{v},\boldsymbol{s})$ is given by

$$\langle\Psi\rangle = \operatorname{tr}\int \Psi(\boldsymbol{x},\boldsymbol{v},\boldsymbol{s})\;\frac{f}{n}\,\mathrm{d}^3v\,. \tag{1.5}$$

$\langle\Psi\rangle$ is real for any Hermitian Ψ as f is Hermitian.

The density of the gas is assumed to be small enough such that two-particle collisions only must be taken into account (dilute gas). In a collision total energy and total momentum of the two particles are conserved, but the orientation of the spins may be changed: spin flip.

Denoting the velocities of two particles before and after a collision by $\boldsymbol{v}_1{}'$, $\boldsymbol{v}_2{}'$ and $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ we have the conservation laws

$$\boldsymbol{v}_1{}' + \boldsymbol{v}_2{}' = \boldsymbol{v}_1 + \boldsymbol{v}_2\,, \tag{1.6}$$

$$v_1{}'^2 + v_2{}'^2 = v_1{}^2 + v_2{}^2. \tag{1.7}$$

The center of mass velocity and the relative velocities are defined by

$$\boldsymbol{v}_\mathrm{s} = \tfrac{1}{2}\,(\boldsymbol{v}_1 + \boldsymbol{v}_2) = \boldsymbol{v}_\mathrm{s}{}', \tag{1.8}$$

$$\boldsymbol{g} = \boldsymbol{v}_1 - \boldsymbol{v}_2;\;\; \boldsymbol{g}' = \boldsymbol{v}_1{}' - \boldsymbol{v}_2{}'. \tag{1.9}$$

Due to (1.7) one has

$$\boldsymbol{g} = g\,\boldsymbol{e}\,,\quad \boldsymbol{g}' = g\,\boldsymbol{e}', \tag{1.10}$$

where $\boldsymbol{e}$, $\boldsymbol{e}'$ are unit vectors. Hence

$$\boldsymbol{v}_{1,2} = \boldsymbol{v}_\mathrm{s} \pm \tfrac{1}{2}\,g\,\boldsymbol{e};\qquad \boldsymbol{v}_{1,2}' = \boldsymbol{v}_\mathrm{s} \pm \tfrac{1}{2}\,g\,\boldsymbol{e}'. \tag{1.11}$$

The relative kinetic energy is

$$E_{12} = \tfrac{1}{2}\,m_{12}\,g^2, \tag{1.12}$$

where $m_{12} = m/2$ is the reduced mass of two particles with mass m.

Assuming the precollisional state to be a plane wave incoming in $\boldsymbol{e}'$ direction in the centre of mass system (where the scattering process is reduced to a one-body problem with the scatterer at the origin $\boldsymbol{r}=0$), one has as the asymptotic solution of the SCHRÖDINGER equation

$$\psi = \left[\exp(i\,k\,r\,\boldsymbol{e}'\cdot\boldsymbol{e}) + \frac{\exp(i\,k\,r)}{r}\,a(\boldsymbol{e},\boldsymbol{e}';\boldsymbol{s}_1,\boldsymbol{s}_2)\right]\chi_1\,\chi_2\,,$$

where r is the distance $r = |\boldsymbol{x}_1 - \boldsymbol{x}_2|$, $\boldsymbol{e}=$ the direc-

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[8] L. WALDMANN, Z. Naturforschg. **18 a**, 1033 [1963].

[9] D. W. CONDIFF, W.-K. LU, and J. S. DAHLER, J. Chem. Phys. **42**, 3445 [1965].

[10] $\varepsilon_{\mu\nu\lambda} = \begin{cases} 1 & \text{if } \mu,\nu,\lambda \text{ even permutation of } 1, 2, 3, \\ -1 & \text{if } \mu,\nu,\lambda \text{ odd permutation of } 1, 2, 3, \\ 0 & \text{otherwise.} \end{cases}$

Throughout this paper we use the summation convention: repeated Greek subscripts have to be summed.

tion of observation and $k = m_{12}\, g/\hbar$. The scattering amplitude a which has the dimension of a length, is a matrix with respect to spin indices operating on the spinors χ_1 and χ_2 of the colliding particles. $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ of course commute. Furthermore the scattering amplitude depends on the relative kinetic energy E_{12}, although this is not indicated in the above notation.

Due to the unitarity of the S-matrix the optical theorem for the scattering amplitude holds:

$$\begin{aligned}\int a(\boldsymbol{e}, \boldsymbol{e}'; \boldsymbol{s}_1, \boldsymbol{s}_2)\, a^\dagger(\boldsymbol{e}', \boldsymbol{e}; \boldsymbol{s}_1, \boldsymbol{s}_2)\, g\, \mathrm{d}^2 e' &= \int a^\dagger(\boldsymbol{e}, \boldsymbol{e}'; \boldsymbol{s}_1, \boldsymbol{s}_2)\, a(\boldsymbol{e}', \boldsymbol{e}; \boldsymbol{s}_1, \boldsymbol{s}_2)\, g\, \mathrm{d}^2 e' \\ &= \frac{h}{i\, m_{12}} [a(\boldsymbol{e}, \boldsymbol{e}; \boldsymbol{s}_1, \boldsymbol{s}_2) - a^\dagger(\boldsymbol{e}, \boldsymbol{e}; \boldsymbol{s}_1, \boldsymbol{s}_2)]\end{aligned} \tag{1.13}$$

where $\mathrm{d}^2 e'$ is the solid angle element and (with magnetic quantum numbers M)

$$a^\dagger(\boldsymbol{e}', \boldsymbol{e})_{M_1'M_2'; M_1M_2} = (a(\boldsymbol{e}, \boldsymbol{e}')_{M_1M_2; M_1'M_2'})^*$$

The transport equation for particles with spin now reads [[1] and [11], p. 484; [12]]

$$\begin{aligned} D_1 f(\boldsymbol{v}_1, \boldsymbol{s}_1) = \mathrm{tr}_2 \int \mathrm{d}^3 v_2 \Big\{ &\int a(\boldsymbol{e}, \boldsymbol{e}'; \boldsymbol{s}_1, \boldsymbol{s}_2)\, f(\boldsymbol{v}_1', \boldsymbol{s}_1)\, f(\boldsymbol{v}_2', \boldsymbol{s}_2)\, a^\dagger(\boldsymbol{e}', \boldsymbol{e}; \boldsymbol{s}_1, \boldsymbol{s}_2)\, g\, \mathrm{d}^2 e' \\ &- \frac{h}{i\, m_{12}} [a(\boldsymbol{e}, \boldsymbol{e}; \boldsymbol{s}_1, \boldsymbol{s}_2)\, f(\boldsymbol{v}_1, \boldsymbol{s}_1)\, f(\boldsymbol{v}_2, \boldsymbol{s}_2) - f(\boldsymbol{v}_1, \boldsymbol{s}_1)\, f(\boldsymbol{v}_2, \boldsymbol{s}_2)\, a^\dagger(\boldsymbol{e}, \boldsymbol{e}; \boldsymbol{s}_1, \boldsymbol{s}_2)] \Big\}. \end{aligned} \tag{1.14}$$

If there is no external magnetic field the differential operator D_1 on the left side of the transport equation has the usual meaning

$$D_1 = D_1^{(0)} \equiv \frac{\partial}{\partial t} + \boldsymbol{v}_1 \cdot \frac{\partial}{\partial \boldsymbol{x}_1}. \tag{1.15}$$

In the presence of a magnetic field with magnitude H and direction $\boldsymbol{h}$ the spin vector $\boldsymbol{s}$ of a free particle will undergo a precessional motion about the unit vector $\boldsymbol{h}$ with the frequency

$$\omega_H = \frac{\mu H}{\hbar S}. \tag{1.16}$$

Herein, μ denotes the magnitude of the magnetic moment which is assumed to be parallel to $\boldsymbol{s}$. In this case one has

$$\begin{aligned} D_1 f(\boldsymbol{v}_1, \boldsymbol{s}_1) &= D_1^{(H)} f(\boldsymbol{v}_1, \boldsymbol{s}_1) \\ &\equiv \left(\frac{\partial}{\partial t} + \boldsymbol{v}_1 \cdot \frac{\partial}{\partial \boldsymbol{x}_1}\right) f(\boldsymbol{v}_1, \boldsymbol{s}_1) \\ &\quad - i\, \omega_H [\boldsymbol{h}\cdot\boldsymbol{s}_1, f(\boldsymbol{v}_1, \boldsymbol{s}_1)] \end{aligned} \tag{1.17}$$

with $$[\boldsymbol{h}\cdot\boldsymbol{s}, f] = \boldsymbol{h}\cdot\boldsymbol{s}\, f - f\, \boldsymbol{h}\cdot\boldsymbol{s}.$$

The commutator describes the free precession of the spin.

The right side of (1.14) is the collision term. The trace operator "tr_2" operates on $\boldsymbol{s}_2$ only. Note that the total collision term contains a term quadratic in the scattering amplitude corresponding to "gain collisions" and a term linear in the forward scattering amplitude corresponding to "loss collisions". Hermiticity of f is conserved in time according to (1.14).

If the particles have no spin – classical case – a and f commute. We may apply the optical theorem and using the cross section

$$\sigma(\boldsymbol{e}, \boldsymbol{e}') = a^\dagger(\boldsymbol{e}, \boldsymbol{e}')\, a(\boldsymbol{e}', \boldsymbol{e})$$

we obtain the classical BOLTZMANN equation

$$D_1^{(0)} f(\boldsymbol{v}_1) = \iint \mathrm{d}^3 v_2 [f(\boldsymbol{v}_1')\, f(\boldsymbol{v}_2') - f(\boldsymbol{v}_1)\, f(\boldsymbol{v}_2)]\, \sigma\, g\, \mathrm{d}^2 e'.$$

In all our considerations we shall use the same collision term in the case with magnetic field as without magnetic field. Thus the energy conservation will not hold exactly if there is a spin flip during a collision. This simplification does not introduce a great error because the magnetic energy is very small compared with the average kinetic energy at room temperature; it is entirely satisfactory for the discussion of the SENFTLEBEN effect, has however to be amended in the discussion of the BLOCH equation for spin resonance.

§ 2. Conservation equations

First we note that the scattering operator a has to be invariant under changing the names of the particles 1,2 → 2,1, i. e.

$$a(\boldsymbol{e}, \boldsymbol{e}'; \boldsymbol{s}_1, \boldsymbol{s}_2) = a(-\boldsymbol{e}, -\boldsymbol{e}'; \boldsymbol{s}_2, \boldsymbol{s}_1). \tag{2.1}$$

[11] L. WALDMANN, Transporterscheinungen in Gasen von mittlerem Druck, in Handbuch der Physik, ed. S. FLÜGGE, Springer, Berlin [1958] vol. 12.

[12] In ref. [1] and [11] a dimensionless "scattering amplitude" a_{dl} was used instead of a. Both are related by
$a_{\mathrm{dl}} = k\, a$ where $k = m_{12}\, g/\hbar$.

The instantaneous local mean value of a quantity $\Psi(\boldsymbol{v}_1, \boldsymbol{s}_1)$ has been defined by (1.5). The time dependence of this mean value is of course governed by the transport equation (1.14) which yields

$$\frac{\partial}{\partial t} n\langle \Psi \rangle + \frac{\partial}{\partial x_\mu} j_\mu^{(\Psi)} = J(\Psi), \tag{2.2}$$

where the Ψ-flux is

$$j_\mu^{(\Psi)} = n\langle v_\mu \Psi \rangle$$

and the local Ψ-production per unit volume is

$$J(\Psi) = \mathrm{tr}_1\,\mathrm{tr}_2 \iint \mathrm{d}^3v_1\,\mathrm{d}^3v_2\,\Psi_1 \Big\{ \int a(\boldsymbol{e},\boldsymbol{e}';\boldsymbol{s}_1,\boldsymbol{s}_2)\, f_1' f_2'\, a^\dagger(\boldsymbol{e}',\boldsymbol{e};\boldsymbol{s}_1,\boldsymbol{s}_2)\, g\,\mathrm{d}^2e' - \frac{h}{i\,m_{12}} [a(\boldsymbol{e},\boldsymbol{e};\boldsymbol{s}_1,\boldsymbol{s}_2)\, f_1 f_2 - f_1 f_2\, a^\dagger(\boldsymbol{e},\boldsymbol{e};\boldsymbol{s}_1,\boldsymbol{s}_2)] \Big\}.$$

Here we use the abbreviations

$$\Psi_1 = \Psi(\boldsymbol{v}_1, \boldsymbol{s}_1); \qquad f_1 = f(\boldsymbol{v}_1, \boldsymbol{s}_1); \qquad f_1' = f(\boldsymbol{v}_1', \boldsymbol{s}_1) \quad \text{etc.}$$

Changing the names of variables 1,2 → 2,1 and using (2.1) and the invariance of $\mathrm{d}^3v_1\,\mathrm{d}^3v_2\,\mathrm{d}^2e'$ under this transformation we obtain

$$J(\Psi) = \tfrac{1}{2}\,\mathrm{tr}_1\,\mathrm{tr}_2 \iint \mathrm{d}^3v_1\,\mathrm{d}^3v_2 (\Psi_1 + \Psi_2) \Big\{ \int a\, f_1' f_2'\, a^\dagger g\,\mathrm{d}^2e' - \frac{h}{i\,m_{12}} [a(0)\, f_1 f_2 - f_1 f_2\, a^\dagger(0)] \Big\}$$

where $a(0)$ stands for the forward scattering amplitude.

Now if Ψ is a spin-free function, it commutes with a and f. Furthermore, the trace of a product of matrices is invariant under a cyclic interchange of the factors. Hence, if Ψ is spin-free we can apply the optical theorem and obtain

$$J(\Psi_{\text{class}}) = \tfrac{1}{2}\,\mathrm{tr}_1\,\mathrm{tr}_2 \iint \mathrm{d}^3v_1\,\mathrm{d}^3v_2 (\Psi_1 + \Psi_2)\, [f_1' f_2'\, a^\dagger(\boldsymbol{e}',\boldsymbol{e};\boldsymbol{s}_1,\boldsymbol{s}_2)\, a(\boldsymbol{e},\boldsymbol{e}';\boldsymbol{s}_1,\boldsymbol{s}_2) - f_1 f_2\, a^\dagger(\boldsymbol{e},\boldsymbol{e}';\boldsymbol{s}_1,\boldsymbol{s}_2)\, a(\boldsymbol{e},\boldsymbol{e}';\boldsymbol{s}_1,\boldsymbol{s}_2)]\, g\,\mathrm{d}^2e'.$$

We now interchange primed and unprimed variables in the gain term and make use of LIOUVILLE's theorem

$$\mathrm{d}^3v_1'\,\mathrm{d}^3v_2'\,\mathrm{d}^2e = \mathrm{d}^3v_1\,\mathrm{d}^3v_2\,\mathrm{d}^2e'. \tag{2.3}$$

This gives

$$J(\Psi_{\text{class}}) = -\tfrac{1}{2}\,\mathrm{tr}_1\,\mathrm{tr}_2 \iint \mathrm{d}^3v_1\,\mathrm{d}^3v_2\,\delta(\Psi)\, f_1 f_2\, a^\dagger(\boldsymbol{e},\boldsymbol{e}';\boldsymbol{s}_1,\boldsymbol{s}_2)\, a(\boldsymbol{e}',\boldsymbol{e};\boldsymbol{s}_1,\boldsymbol{s}_2)\, g\,\mathrm{d}^2e'$$

with

$$\delta(\Psi) = \Psi_1 + \Psi_2 - \Psi_1' - \Psi_2'. \tag{2.4}$$

Equation (2.4) is valid for any spin-free Ψ. But for $\Psi = 1,\ v_\mu,\ v^2$ one has according to the conservation laws in binary collisions $\delta(\Psi) = 0$ and thus $J(\Psi) = 0$. This will give us the conservation equations for particle number, momentum and energy.

§ 3. *H-theorem. Thermal equilibrium*

The validity of the H-theorem for particles with spin has previously been proved by WALDMANN (s. [11], p. 484). Here we shall give a short review of these results:

a) Unitary transformation of transport equation

Here we shall use a matrix notation with respect to spin indices omitting the variable $\boldsymbol{s}$. Similarly we omit the variables t and $\boldsymbol{x}$. $U(\boldsymbol{v})$ [more fully $U(t, \boldsymbol{x}, \boldsymbol{v})$] shall be a unitary matrix with respect to spin indices depending on $\boldsymbol{x}$, $\boldsymbol{v}$ as parameters, $UU^\dagger = U^\dagger U = 1$. The distribution matrix may be transformed according to

$$\hat{f}(\boldsymbol{v}) = U^\dagger(\boldsymbol{v})\, f(\boldsymbol{v})\, U(\boldsymbol{v}). \tag{3.1}$$

The direct product $f_{12} = f_1 f_2$ then transforms after

$$\hat{f}(\boldsymbol{v}) = U^\dagger(\boldsymbol{v}_1, \boldsymbol{v}_2)\, f_{12}\, U(\boldsymbol{v}_1, \boldsymbol{v}_2) \tag{3.1a}$$

where $U(\boldsymbol{v}_1, \boldsymbol{v}_2) = U(\boldsymbol{v}_1)\, U(\boldsymbol{v}_2)$. Likewise we introduce a transformed scattering amplitude $\hat{a}$ pertaining to two particles with $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ by

$$\begin{aligned} &\hat{a}(\boldsymbol{v}_1, \boldsymbol{v}_2; \boldsymbol{v}_1', \boldsymbol{v}_2') \\ &\quad = U^\dagger(\boldsymbol{v}_1, \boldsymbol{v}_2)\, a(\boldsymbol{e}, \boldsymbol{e}')\, U(\boldsymbol{v}_1', \boldsymbol{v}_2'), \\ &\hat{a}^\dagger(\boldsymbol{v}_1', \boldsymbol{v}_2'; \boldsymbol{v}_1, \boldsymbol{v}_2) \\ &\quad = U^\dagger(\boldsymbol{v}_1', \boldsymbol{v}_2')\, a^\dagger(\boldsymbol{e}', \boldsymbol{e})\, U(\boldsymbol{v}_1, \boldsymbol{v}_2). \end{aligned} \tag{3.2}$$

The transformed transport equation then reads

$$D_1 \hat{f}(\boldsymbol{v}_1) + [U^\dagger(\boldsymbol{v}_1)\, D_1 U(\boldsymbol{v}_1), \hat{f}(\boldsymbol{v}_1)] = \mathrm{tr}_2 \int \mathrm{d}^3 v_2 \, \Big\{ \int \hat{a}(\boldsymbol{v}_1, \boldsymbol{v}_2; \boldsymbol{v}_1', \boldsymbol{v}_2')\, \hat{f}(\boldsymbol{v}_1', \boldsymbol{v}_2')\, \hat{a}^\dagger(\boldsymbol{v}_1', \boldsymbol{v}_2'; \boldsymbol{v}_1, \boldsymbol{v}_2)\, g\, \mathrm{d}^2 e'$$
$$- \frac{h}{i\, m_{12}} [\hat{a}(\boldsymbol{v}_1, \boldsymbol{v}_2; \boldsymbol{v}_1, \boldsymbol{v}_2)\, \hat{f}(\boldsymbol{v}_1, \boldsymbol{v}_2) - \hat{f}(\boldsymbol{v}_1, \boldsymbol{v}_2)\, \hat{a}^\dagger(\boldsymbol{v}_1, \boldsymbol{v}_2; \boldsymbol{v}_1, \boldsymbol{v}_2)] \Big\}. \tag{3.3}$$

Now, it is always possible to choose a transformation $U(t, \boldsymbol{x}, \boldsymbol{v})$ such that $\hat{f}$ is diagonal with respect to spin indices for all times, space points and velocities:

$$\hat{f}_{M_1 M_1'}(\boldsymbol{v}_1) = F_M(\boldsymbol{v}_1)\, \delta_{M_1 M_1'} \geqq 0\,. \tag{3.4}$$

[In order to give such a U explicitly we first would have to solve (1.14)].

The diagonal elements of the commutator on the left side of (3.3) vanish for a diagonal f and we get

$$\frac{\partial F_{M_1}(\boldsymbol{v}_1)}{\partial t} + \boldsymbol{v}_1 \cdot \frac{\partial}{\partial \boldsymbol{x}_1} F_{M_1}(\boldsymbol{v}_1)$$
$$= \sum_{M_2} \int \mathrm{d}^3 v_2 \Big\{ \int \sum_{M_1' M_2'} F_{M_1'}(\boldsymbol{v}_1')\, F_{M_2'}(\boldsymbol{v}_2')\, A_{M_1 M_2; M_1' M_2'}(\boldsymbol{v}_1, \boldsymbol{v}_2; \boldsymbol{v}_1', \boldsymbol{v}_2')\, A^\dagger_{M_1' M_2'; M_1 M_2}(\boldsymbol{v}_1', \boldsymbol{v}_2'; \boldsymbol{v}_1, \boldsymbol{v}_2)\, g\, \mathrm{d}^2 e'$$
$$- F_{M_1}(\boldsymbol{v}_1)\, F_{M_2}(\boldsymbol{v}_2) \frac{h}{i\, m_{12}} [(A(\boldsymbol{v}_1, \boldsymbol{v}_2; \boldsymbol{v}_1, \boldsymbol{v}_2) - A^\dagger(\boldsymbol{v}_1, \boldsymbol{v}_2; \boldsymbol{v}_1, \boldsymbol{v}_2))_{M_1 M_2; M_1' M_2'}] \Big\}.$$

Here we wrote A instead of $\hat{a}$ in order to indicate that we have used a special transformation.

We now apply the optical theorem which is invariant with respect to the above unitary transformation (s. [11], p. 473). Using the cross section

$$\sigma_{M_1' M_2'; M_1 M_2}(\boldsymbol{v}_1', \boldsymbol{v}_2'; \boldsymbol{v}_1, \boldsymbol{v}_2) = |A_{M_1 M_2; M_1' M_2'}(\boldsymbol{v}_1, \boldsymbol{v}_2; \boldsymbol{v}_1', \boldsymbol{v}_2')|^2$$

we finally obtain

$$\frac{\partial}{\partial t} F_{M_1}(\boldsymbol{v}_1) + \boldsymbol{v}_1 \cdot \frac{\partial}{\partial \boldsymbol{x}_1} F_{M_1}(\boldsymbol{v}_1) \tag{3.5}$$
$$= \sum_{M_2 M_1' M_2'} \iint [F_{M_1'}(\boldsymbol{v}_1')\, F_{M_2'}(\boldsymbol{v}_2') - F_{M_1}(\boldsymbol{v}_1)\, F_{M_2}(\boldsymbol{v}_2)]\, \sigma_{M_1' M_2'; M_1 M_2}(\boldsymbol{v}_1', \boldsymbol{v}_2'; \boldsymbol{v}_1, \boldsymbol{v}_2)\, g\, \mathrm{d}^2 e'\, \mathrm{d}^3 v_2\,.$$

b) Entropy and H-theorem

For the local entropy density we make the ansatz

$$\varrho\, s = -k \sum_{M_1} \int F_{M_1}(\boldsymbol{v}_1) \ln F_{M_1}(\boldsymbol{v}_1)\, \mathrm{d}^3 v_1; \tag{3.6}$$

ϱ density, k BOLTZMANN's constant.

The change in time of s due to collisions is

$$\varrho \left(\frac{\delta s}{\delta t}\right)_{\mathrm{coll}} = -k \sum_{M_1} [\ln F_{M_1} + 1] \left(\frac{\delta F_{M_1}}{\delta t}\right)_{\mathrm{coll}}.$$

On account of particle conservation the second term will vanish and we are left with

$$\varrho \left(\frac{\delta s}{\delta t}\right)_{\mathrm{coll}} = -k \sum_{\substack{M_1 M_2 \\ M_1' M_2'}} \iiint \ln F_{M_1} [F_{M_1'} F_{M_2'} - F_{M_1} F_{M_2}]\, \sigma_{M_1' M_2'; M_1 M_2}\, g\, \mathrm{d}^2 e'\, \mathrm{d}^3 v_1\, \mathrm{d}^3 v_2\,.$$

We change names of particles 1,2 → 2,1 and obtain

$$\varrho \left(\frac{\delta s}{\delta t}\right)_{\mathrm{coll}} = -\tfrac{1}{2} k \sum_{M_1 \ldots M_2'} \iiint \ln(F_{M_1} F_{M_2}) [F_{M_2'} F_{M_1'} - F_{M_1} F_{M_2}]\, \sigma\, g\, \mathrm{d}^2 e'\, \mathrm{d}^3 v_1\, \mathrm{d}^3 v_2\,.$$

Then we exchange primed and unprimed variables in the second term and make use of LIOUVILLE's theorem, finding

$$\varrho \left(\frac{\delta s}{\delta t}\right)_{\mathrm{coll}} = -\tfrac{1}{2} k \sum_{M_1 \ldots M_2'} \iiint \ln\left(\frac{F_{M_1} F_{M_2}}{F_{M_1'} F_{M_2'}}\right) F_{M_1'} F_{M_2'}\, \sigma_{M_1' M_2'; M_1 M_2}\, g\, \mathrm{d}^2 e'\, \mathrm{d}^3 v_1\, \mathrm{d}^3 v_2\,. \tag{3.7}$$

For particles with spin in contrast to monatomic gases no detailed balance symmetry exists; in general one has

$$\sigma_{M_1'M_2';M_1M_2} \neq \sigma_{M_1M_2;M_1'M_2'}\,.$$

Nevertheless one can prove the H-theorem simply by using the first equality of the optical theorem (1.13). In terms of σ the diagonal element $M_1\,M_2$ of this equality reads

$$\sum_{M_1'M_2'} \int \sigma_{M_1M_2;M_1'M_2'}\,(\boldsymbol{v}_1, \boldsymbol{v}_2;\ \boldsymbol{v}_1', \boldsymbol{v}_2')\ g\,\mathrm{d}^2e' = \sum_{M_1'M_2'} \int \sigma_{M_1'M_2';M_1M_2}\,(\boldsymbol{v}_1', \boldsymbol{v}_2';\ \boldsymbol{v}_1, \boldsymbol{v}_2)\ g\,\mathrm{d}^2e'$$

(equality of "right and left normalisation"). We multiply this equation by $F_{M_1}\,(\boldsymbol{v}_1)\ F_{M_2}(\boldsymbol{v}_2)$, integrate over $\mathrm{d}^3v_1\,\mathrm{d}^3v_2$ and sum over M_1, M_2. Then we exchange primed and unprimed variables on the left side and make use of LIOUVILLE's theorem. Thus we obtain the identity

$$\sum_{\substack{M_1M_2\\M_1'M_2'}} \iiint [F_{M_1'}\,(\boldsymbol{v}_1')\ F_{M_2'}\,(\boldsymbol{v}_2') - F_{M_1}\,(\boldsymbol{v}_1)\ F_{M_2}\,(\boldsymbol{v}_2)]\ \sigma_{M_1'M_2';M_1M_2}\ g\,\mathrm{d}^2e'\,\mathrm{d}^3v_1\,\mathrm{d}^3v_2 = 0\,. \qquad (3.8)$$

We now multiply equ. (3.8) by $-k/2$ and add this identity to equation (3.7), obtaining

$$\varrho\left(\frac{\delta s}{\delta t}\right)_{\mathrm{coll}} = \tfrac{1}{2}k \sum_{\substack{M_1M_2\\M_1'M_2'}} \iiint \left[\frac{F_{M_1}F_{M_2}}{F_{M_1'}F_{M_2'}} - 1 - \ln\left(\frac{F_{M_1}F_{M_2}}{F_{M_1'}F_{M_2'}}\right)\right] F_{M_1'}\,F_{M_2'}\ \sigma_{M_1'M_2';M_1M_2}\,g\,\mathrm{d}^2e'\,\mathrm{d}^3v_1\,\mathrm{d}^3v_2\,. \qquad (3.9)$$

But F and σ are non-negative throughout and

$$x - 1 - \ln x > 0\ \ (=0)\ \text{ for positive } x \neq 1\ \ (=1).$$

Thus we see that the integrand of (3.9) is non-negative und we have proved the H-theorem:

$$(\delta s/\delta t)_{\mathrm{coll}} \geqq 0. \qquad (3.10)$$

c) Thermal equilibrium

The entropy of an isolated system increases until it reaches a maximum value: thermal equilibrium.

Then $(\delta s/\delta t)_{\mathrm{coll,\,equil.}} = 0$ and this is true, if and only if

$$F_{M_1}\,(\boldsymbol{v}_1)\ F_{M_2}\,(\boldsymbol{v}_2) = F_{M_1'}\,(\boldsymbol{v}_1')\ F_{M_2'}\,(\boldsymbol{v}_2') \qquad (3.11)$$

for all values of $\boldsymbol{v}_1'\,\boldsymbol{v}_2';\ \boldsymbol{v}_1\,\boldsymbol{v}_2;\ M_1'\,M_2';\ M_1\,M_2$ which are allowed according to the conservation laws (have a $\sigma \neq 0$).

Without a magnetic field the energy does not depend on the magnetic quantum number, thus $F_{M'}(\boldsymbol{v}') = F(\boldsymbol{v}')$ where $F(\boldsymbol{v}')$ can depend on conserved quantities only. In thermal equilibrium we have

$$[f_{MM'}(\boldsymbol{v})]_{\mathrm{equ}} = \frac{n}{2\,S+1}\left(\frac{m}{2\,\pi\,k\,T}\right)^{3/2} \exp\left\{-\frac{m}{2\,k\,T}\,(\boldsymbol{v} - \langle\boldsymbol{v}\rangle)^2\right\}\,\delta_{MM'}\,. \qquad (3.12)$$

Due to the factor $\delta_{MM'}$ the spins are oriented isotropically in space. The normalisation constant in (3.12) has been chosen such that

$$\mathrm{tr}\int \mathrm{d}^3v\,f = n \ \text{ as }\ \ \mathrm{tr}\,1 = 2\,S+1\,.$$

B) Linearized Collision Operator

§ 4. *Linearization of the collision term*

Returning to the general non-equilibrium case we make the ansatz for the distribution

$$f = f_0(1 + \Phi) \qquad (4.1)$$

with

$$f_0 = \frac{1}{2\,S+1}\,n_0\left(\frac{m}{2\,\pi\,k\,T_0}\right)^{3/2} \exp\left\{-\frac{m}{2\,k\,T_0}\,v^2\right\}. \qquad (4.2)$$

f_0 is a standard equilibrium distribution with particle density n_0 and temperature T_0. The pertaining mean velocity $\langle\boldsymbol{v}\rangle_0$ has been taken equal to zero; this simply means a convenient choice of the reference system and doesn't involve a loss of generality. Since the magnetic energy $\omega_{\mathrm{H}}\,\hbar$ is very small compared with $k\,T_0$ we use the same f_0 for the case with magnetic field. Φ measures the deviation of f from this equilibrium state.

The mean value of an operator $\Psi(\boldsymbol{v}, \boldsymbol{s})$ in the equilibrium state is given by

$$\langle\Psi\rangle_0 = \mathrm{tr}\int \Psi(\boldsymbol{v}, \boldsymbol{s})\ \frac{f_0}{n_0}\,\mathrm{d}^3v\,. \qquad (4.3)$$

Further, the obvious properties of f_0 are stated

$$f_0(\boldsymbol{v}) = f_0(-\boldsymbol{v})$$

$$f_0(\boldsymbol{v}_1)\ f_0(\boldsymbol{v}_2) = f_0(\boldsymbol{v}_1')\ f_0(\boldsymbol{v}_2'). \qquad (4.4)$$

The second equality holds on account of the energy conservation.

The product of two one-particle functions occurring in the colilsion term of (1.14) reads

$$f_1 f_2 = f_{01} f_{02} [1 + \Phi_1 + \Phi_2 + \Phi_1 \Phi_2]. \tag{4.5}$$

We assume that the deviation of the system from the standard equilibrium state f_0 is small such that we can neglect the quadratic term $\Phi_1 \Phi_2$. In this approximation we obtain from (1.14)

$$D(\Phi)_1 + \omega(\Phi)_1 = 0. \tag{4.6}$$

Here, according to (1.17), one has

$$D(\Phi)_1 \equiv \left(\frac{\partial \Phi}{\partial t} + \boldsymbol{v} \cdot \frac{\partial \Phi}{\partial \boldsymbol{x}}\right)_1 - i\,\omega_H [\boldsymbol{h}\cdot\boldsymbol{s}_1\, \Phi_1 - \Phi_1\, \boldsymbol{h}\cdot\boldsymbol{s}_1] \tag{4.7}$$

and the linearized collision operator

$$\begin{aligned} \omega(\Phi)_1 = {} & -\mathrm{tr}_2 \int \mathrm{d}^3 v_2\, f_{02} \\ & \cdot \Big\{ \int a(\boldsymbol{e}, \boldsymbol{e}'; \boldsymbol{s}_1, \boldsymbol{s}_2)\, \Phi_{12}'\, a^\dagger(\boldsymbol{e}', \boldsymbol{e}; \boldsymbol{s}_1, \boldsymbol{s}_2)\, g\, \mathrm{d}^2 e' \\ & - \frac{h}{i\, m_{12}} [a(\boldsymbol{e}, \boldsymbol{e}; \boldsymbol{s}_1, \boldsymbol{s}_2)\, \Phi_{12} - \Phi_{12}\, a^\dagger(\boldsymbol{e}, \boldsymbol{e}; \boldsymbol{s}_1, \boldsymbol{s}_2)] \Big\} \end{aligned} \tag{4.8}$$

has been introduced with the abbreviations

$$\Phi_{12} = \Phi_1 + \Phi_2; \quad \Phi_{12}' = \Phi_1' + \Phi_2'.$$

From the definition of the collision operator ω we see that $[\omega(\Phi)]^\dagger = \omega(\Phi^\dagger)$. Hence $\omega(\Phi)$ is Hermitian if Φ itself is so:

$$[\omega(\Phi)]^\dagger = \omega(\Phi). \tag{4.9}$$

Again f will remain Hermitian if it is so at time $t = 0$. The mean value of a Hermitian Φ formed according to (1.5) will then be real at any time.

If Φ is a spin-free function the collision term can be simplified by using the optical theorem

$$\begin{aligned} \omega(\Phi)_1 = {} & \iint f_{02} (\Phi_{12} - \Phi_{12}') \\ & \cdot \mathrm{tr}_2 [a(\boldsymbol{e}, \boldsymbol{e}'; \boldsymbol{s}_1, \boldsymbol{s}_2)\, a^\dagger(\boldsymbol{e}', \boldsymbol{e}; \boldsymbol{s}_1, \boldsymbol{s}_2)]\, g\, \mathrm{d}^2 e'. \end{aligned} \tag{4.10}$$

Due to the conservation laws one has

$$\omega(1) = 0, \quad \omega(v^2) = 0, \quad \omega(v_\mu) = 0. \tag{4.11}$$

§ 5. Collision brackets

We define the collision bracket of two operators Ψ, Φ by

$$\langle \Psi\, \omega(\Phi) \rangle_0 = \mathrm{tr}_1 \int \frac{f_{01}}{n_0}\, \Psi_1\, \omega(\Phi)_1\, \mathrm{d}^3 v_1. \tag{5.1}$$

The complex conjugate of the collision bracket is defined by

$$\langle \Psi\, \omega(\Phi) \rangle_0^* = \langle \Psi^\dagger\, \omega(\Phi)^\dagger \rangle_0.$$

If Ψ, Φ are Hermitian, one has after (4.9)

$$\langle \Psi\, \omega(\Phi) \rangle_0^* = \langle \Psi\, \omega(\Phi) \rangle_0, \tag{5.2}$$

the collision bracket is real.

Now we shall show some further general properties of the collision bracket.

a) The positive semi-definiteness of the collision bracket

In order to show that the collision bracket of a Hermitian operator Ψ with itself is non-negative we only need the trivial symmetry (2.1) of the scattering operator with respect to interchange of names of particles and the optical theorem (1.13).

Using (1.13) and interchanging names of particles 1,2 → 2,1 we obtain

$$\begin{aligned} \langle \Psi\, \omega(\Psi) \rangle_0 = \tfrac{1}{2}\, \mathrm{tr}_1\, \mathrm{tr}_2 \int \frac{1}{n_0} f_{01} f_{02}\, \Psi_{12} \{ & -a(\boldsymbol{e}, \boldsymbol{e}')\, \Psi_{12}'\, a^\dagger(\boldsymbol{e}', \boldsymbol{e}) \\ & + a(\boldsymbol{e}, \boldsymbol{e}')\, a^\dagger(\boldsymbol{e}', \boldsymbol{e})\, \Psi_{12} \}\, g\, \mathrm{d}^2 e'\, \mathrm{d}^3 v_1\, \mathrm{d}^3 v_2. \end{aligned}$$

Now we use the cyclic symmetry of $\mathrm{tr}_1\, \mathrm{tr}_2$ applied to a product of operators:

$$\begin{aligned} \langle \Psi\, \omega(\Psi) \rangle_0 = \tfrac{1}{4}\, \mathrm{tr}_1\, \mathrm{tr}_2 \int \frac{1}{n_0} f_{01} f_{02} \{ & -\Psi_{12}\, a(\boldsymbol{e}, \boldsymbol{e}')\, \Psi_{12}'\, a^\dagger(\boldsymbol{e}', \boldsymbol{e}) + \Psi_{12}\, a(\boldsymbol{e}, \boldsymbol{e}')\, a^\dagger(\boldsymbol{e}', \boldsymbol{e})\, \Psi_{12} \\ & - a(\boldsymbol{e}, \boldsymbol{e}')\, \Psi_{12}'\, a^\dagger(\boldsymbol{e}', \boldsymbol{e})\, \Psi_{12} + a(\boldsymbol{e}, \boldsymbol{e}')\, a^\dagger(\boldsymbol{e}', \boldsymbol{e})\, \Psi_{12}\, \Psi_{12} \}\, g\, \mathrm{d}^2 e'\, \mathrm{d}^3 v_1\, \mathrm{d}^3 v_2. \end{aligned}$$

According to the first equality of (1.13) the last term is equal to

$$\tfrac{1}{4}\, \mathrm{tr}_1\, \mathrm{tr}_2 \int \frac{1}{n_0} f_{01} f_{02}\, a(\boldsymbol{e}', \boldsymbol{e})\, \Psi_{12}\, \Psi_{12}\, a^\dagger(\boldsymbol{e}, \boldsymbol{e}')\, g\, \mathrm{d}^2 e'\, \mathrm{d}^3 v_1\, \mathrm{d}^3 v_2.$$

Finally we exchange primed and unprimed variables in this term, using Liouville's theorem. With the abbreviations

$$A_{12} = \Psi_{12}\, a(\boldsymbol{e}, \boldsymbol{e}') - a(\boldsymbol{e}, \boldsymbol{e}')\, \Psi_{12}'; \quad A_{12}^{\dagger} = a^{\dagger}(\boldsymbol{e}', \boldsymbol{e})\, \Psi_{12} - \Psi_{12}'\, a^{\dagger}(\boldsymbol{e}', \boldsymbol{e}),$$

we get

$$\langle \Psi\, \omega(\Psi) \rangle_0 = \tfrac{1}{4} \int \frac{1}{n_0} f_{01}\, f_{02}\, \mathrm{tr}_1\, \mathrm{tr}_2 (A_{12}\, A_{12}^{\dagger})\;\; g\, \mathrm{d}^2 e'\, \mathrm{d}^3 v_1\, \mathrm{d}^3 v_2 \geqq 0\,. \tag{5.3}$$

As the integrand is non-negative one has the desired property

$$\langle \Psi\, \omega(\Psi) \rangle_0 \geqq 0\,.$$

This collision bracket is 0 if and only if $A_{12} = A_{12}^{\dagger} = 0$ or

$$\Psi_{12}\, a = a\, \Psi_{12}'.$$

b) The corresponding collision operator

In order to investigate the consequence of exchanging two operators Φ, Ψ in the collision bracket we define a "corresponding" collision operator $\tilde{\omega}$ by [3]

$$\tilde{\omega}(\Psi)_1 = -\,\mathrm{tr}_2 \int \mathrm{d}^3 v_2\, f_{02} \left\{ \int a^{\dagger}(\boldsymbol{e}, \boldsymbol{e}'; \boldsymbol{s}_1, \boldsymbol{s}_2)\;\; \Psi_{12}'\, a(\boldsymbol{e}', \boldsymbol{e}; \boldsymbol{s}_1, \boldsymbol{s}_2)\, g\, \mathrm{d}^2 e' \right. \\ \left. + \frac{h}{i\, m_{12}} \left[a^{\dagger}(\boldsymbol{e}, \boldsymbol{e}; \boldsymbol{s}_1, \boldsymbol{s}_2)\;\; \Psi_{12} - \Psi_{12}\, a(\boldsymbol{e}, \boldsymbol{e}; \boldsymbol{s}_1, \boldsymbol{s}_2) \right] \right\}. \tag{5.4}$$

If one explicitly writes down $\langle \Phi\, \tilde{\omega}(\Psi) \rangle_0$, exchanges primed and unprimed variables in the first term using Liouville's theorem, then exchanges the names of particles 1,2 in all terms containing Ψ_2 and makes use of the cyclic symmetry of the trace one finds the interchange property

$$\langle \Phi\, \tilde{\omega}(\Psi) \rangle_0 = \langle \Psi\, \omega(\Phi) \rangle_0\,. \tag{5.5}$$

Note that $\tilde{\omega} \neq \omega$, in contrast to the monatomic gas.

c) Parity invariance

One says the parity is conserved during a collision if the scattering operator a is invariant in the following sense (P-invariant)

$$a(\boldsymbol{e}, \boldsymbol{e}'; \boldsymbol{s}_1, \boldsymbol{s}_2) = a(-\boldsymbol{e}, -\boldsymbol{e}'; \boldsymbol{s}_1, \boldsymbol{s}_2)\,. \tag{5.6}$$

For any operator Ψ we define another operator Ψ_P, the "mirrored" one, by

$$\Psi_P(\boldsymbol{v}, \boldsymbol{s}) = \Psi(-\boldsymbol{v}, \boldsymbol{s})\,. \tag{5.7}$$

If

$$\Psi_P = \pm\, \Psi \tag{5.8}$$

one says Ψ has even (+) or odd (−) parity.

Writing down the collision bracket $\langle \Psi\, \omega(\Phi) \rangle_0$ explicitly, changing integration variables in the way

$$\boldsymbol{v}_1, \boldsymbol{v}_2; \boldsymbol{v}_1', \boldsymbol{v}_2' \to -\boldsymbol{v}_1,\, -\boldsymbol{v}_2;\, -\boldsymbol{v}_1',\, -\boldsymbol{v}_2'; \\ \boldsymbol{e}, \boldsymbol{e}' \to -\boldsymbol{e},\, -\boldsymbol{e}'$$

and using the P-invariance of the scattering amplitude (5.6) and the definition (5.7) we see that

$$\langle \Psi\, \omega(\Phi) \rangle_0 = \langle \Psi_P\, \omega(\Phi_P) \rangle_0\,. \tag{5.9}$$

Due to this identity the collision bracket of two operators with different parity will vanish [3].

d) Time reversal invariance

Now we assume the scattering operator to be invariant under time reversal or T-invariant which means

$$a(\boldsymbol{e}, \boldsymbol{e}'; \overrightarrow{\boldsymbol{s}_1}, \overrightarrow{\boldsymbol{s}_2}) = a(-\boldsymbol{e}', -\boldsymbol{e}; -\overleftarrow{\boldsymbol{s}_1}, -\overleftarrow{\boldsymbol{s}_2})\,. \tag{5.10}$$

The scattering amplitude has to be considered as a polynomial in the components of $\boldsymbol{s}_{1,2}$. The arrow pointing to the left indicates that in products of components of $\boldsymbol{s}_{1,2}$ the factors have to be taken in the reverse order compared with the original function indicated by arrows pointing to the right.

For any operator Ψ we define a Ψ_T, the "time reversed" one, by

$$\Psi_T(\boldsymbol{v}, \overrightarrow{\boldsymbol{s}}) = \Psi(-\boldsymbol{v}, -\overleftarrow{\boldsymbol{s}})\,. \tag{5.11}$$

If

$$\Psi_T = \pm\, \Psi \tag{5.12}$$

one says Ψ is even (+) or odd (−) under time reversal. Obviously, for a product of two operators $\Phi\, \Psi$ one has

$$(\Phi\, \Psi)_T = \Psi_T\, \Phi_T\,. \tag{5.13}$$

Furthermore, from the invariance of the commutation rule (1.1) under time reversal (i. e. changing sign of $\boldsymbol{s}$ and order of factors), it follows that

$$\mathrm{tr}\, F(\overrightarrow{\boldsymbol{s}}) = \mathrm{tr}\, F(-\overleftarrow{\boldsymbol{s}}) \tag{5.14}$$

where F is any function of $\boldsymbol{s}$.

In order to study the behaviour of the collision bracket under time reversal we write down $\langle \Phi\, \tilde{\omega}(\Psi) \rangle_0$ according to (5.4), at the same time using (5.14)

$$\langle \Phi\, \tilde{\omega}(\Psi) \rangle_0 = -\mathrm{tr}_1\, \mathrm{tr}_2 \iint \mathrm{d}^3v_1\, \mathrm{d}^3v_2\, \frac{1}{n_0}\, f_{01}\, f_{02} \Big\{ \int a(\boldsymbol{e}', \boldsymbol{e};\, -\overleftarrow{\boldsymbol{s}}_1,\, -\overleftarrow{\boldsymbol{s}}_2)\; \Psi_{12}{}'(-\overleftarrow{\boldsymbol{s}}_1,\, -\overleftarrow{\boldsymbol{s}}_2)\; a^\dagger(\boldsymbol{e}, \boldsymbol{e}';\, -\overleftarrow{\boldsymbol{s}}_1,\, -\overleftarrow{\boldsymbol{s}}_2)\; \Phi(-\overleftarrow{\boldsymbol{s}}_1)\, g\, \mathrm{d}^2e'$$
$$- \frac{h}{i\, m_{12}} [a(\boldsymbol{e}, \boldsymbol{e};\, -\overleftarrow{\boldsymbol{s}}_1,\, -\overleftarrow{\boldsymbol{s}}_2)\;\, \Psi_{12}(-\overleftarrow{\boldsymbol{s}}_1,\, -\overleftarrow{\boldsymbol{s}}_2) - \Psi_{12}(-\overleftarrow{\boldsymbol{s}}_1,\, -\overleftarrow{\boldsymbol{s}}_2)\; a^\dagger(\boldsymbol{e}, \boldsymbol{e};\, -\overleftarrow{\boldsymbol{s}}_1,\, -\overleftarrow{\boldsymbol{s}}_2)]\; \Phi(-\overleftarrow{\boldsymbol{s}}_1) \Big\}.$$

Now we change integration variables according to

$$\boldsymbol{v}_1, \boldsymbol{v}_2; \boldsymbol{v}_1{}', \boldsymbol{v}_2{}' \to -\boldsymbol{v}_1, -\boldsymbol{v}_2; -\boldsymbol{v}_1{}', -\boldsymbol{v}_2{}';$$
$$\boldsymbol{e}, \boldsymbol{e}' \to -\boldsymbol{e}, -\boldsymbol{e}',$$

use the T-invariance of the scattering operator and the definition (5.11). This gives

$$\langle \Phi\, \tilde{\omega}(\Psi) \rangle_0 = \langle \Phi_T\, \omega(\Psi_T) \rangle_0 . \qquad (5.15)$$

But according to (5.5) in any case the $\tilde{\omega}$-bracket can be replaced by a corresponding ω-bracket, yielding

$$\langle \Psi\, \omega(\Phi) \rangle_0 = \langle \Phi_T\, \omega(\Psi_T) \rangle_0 . \qquad (5.16)$$

This important relation expresses the time-reversibility of the binary scattering amplitude in terms of the collision brackets. It will give us the ONSAGER–CASIMIR symmetry relations for the transport coefficients [3].

e) Isotropy

The collision operator ω is invariant with respect to a rotation of the coordinate system. Thus, the operator ω will map a Φ which is an irreducible tensor of rank l into a tensor of the same kind.

§ 6. *Symmetries of* $\boldsymbol{v}$- *and* $\boldsymbol{s}$-*brackets*

In the part C we shal expand the linearized transport equation in a series of certain basic functions which are then eliminated. In this process the differential part $D(\Phi)_1$ leads to $\boldsymbol{v}$- and $\boldsymbol{s}$-brackets of the following type

$$\langle \Psi \,|\, \boldsymbol{v} \,|\, \Phi \rangle_0 = \mathrm{tr} \int \frac{f_0}{n_0}\, \Psi\, \boldsymbol{v}\, \Phi\, \mathrm{d}^3v\,, \qquad (6.1)$$

$$\langle \Psi \,|\, \boldsymbol{s} \,|\, \Phi \rangle_0 = \mathrm{tr} \int \frac{f_0}{n_0}\, \Psi [\boldsymbol{s}\, \Phi - \Phi\, \boldsymbol{s}]\, \mathrm{d}^3v\,. \qquad (6.2)$$

We ask for their general properties.

a) $\boldsymbol{v}$-brackets

Because $\boldsymbol{v}$ does not depend on the spin and due to the cyclic symmetry of the trace one has

$$\langle \Psi \,|\, \boldsymbol{v} \,|\, \Phi \rangle_0 = \langle \Phi \,|\, \boldsymbol{v} \,|\, \Psi \rangle_0 . \qquad (6.3)$$

Changing the integration variable $\boldsymbol{v}$ to $-\boldsymbol{v}$ and using the definition (5.7) we see that

$$\langle \Psi \,|\, \boldsymbol{v} \,|\, \Phi \rangle_0 = -\langle \Psi_P \,|\, \boldsymbol{v} \,|\, \Phi_P \rangle_0 . \qquad (6.4)$$

Thus the $\boldsymbol{v}$-bracket will vanish for two operators Ψ, Φ with equal parity.

Using (5.13, 14), changing the variable $\boldsymbol{v}$ to $-\boldsymbol{v}$ and using the definition (5.11) we obtain

$$\langle \Psi \,|\, \boldsymbol{v} \,|\, \Phi \rangle_0 = -\langle \Phi_T \,|\, \boldsymbol{v} \,|\, \Psi_T \rangle_0 . \qquad (6.5)$$

Furthermore according to (6.3) one can interchange Φ_T and Ψ_T on the right hand side:

$$\langle \Psi \,|\, \boldsymbol{v} \,|\, \Phi \rangle_0 = -\langle \Psi_T \,|\, \boldsymbol{v} \,|\, \Phi_T \rangle_0 . \qquad (6.6)$$

Thus the $\boldsymbol{v}$-matrix element will vanish for two operators with equal behaviour under time reversal.

b) $\boldsymbol{s}$-brackets

Due to the cyclic invariance of tr one has

$$\langle \Psi \,|\, \boldsymbol{s} \,|\, \Phi \rangle_0 = \mathrm{tr} \int \frac{f_0}{n_0} [\Phi\, \Psi\, \boldsymbol{s} - \Phi\, \boldsymbol{s}\, \Psi]\, \mathrm{d}^3v$$
$$= -\langle \Phi \,|\, \boldsymbol{s} \,|\, \Psi \rangle_0 . \qquad (6.7)$$

If we change the integration variable $\boldsymbol{v}$ to $-\boldsymbol{v}$ and use definition (5.7), we see that

$$\langle \Psi \,|\, \boldsymbol{s} \,|\, \Phi \rangle_0 = \langle \Psi_P \,|\, \boldsymbol{s} \,|\, \Phi_P \rangle_0 . \qquad (6.8)$$

Thus the $\boldsymbol{s}$-bracket vanishes for operators Φ, Ψ with different parities.

Applying (5.14) and changing variable $\boldsymbol{v}$ to $-\boldsymbol{v}$ one has

$$\langle \Psi \,|\, \boldsymbol{s} \,|\, \Phi \rangle_0 = -\mathrm{tr} \int \frac{f_0}{n_0} [\Phi_T\, \boldsymbol{s}\, \Psi_T - \boldsymbol{s}\, \Phi_T\, \Psi_T]\, \mathrm{d}^3v$$
$$= -\langle \Phi_T \,|\, \boldsymbol{s} \,|\, \Psi_T \rangle_0 .$$

According to (6.7) one can rewrite

$$\langle \Psi \,|\, \boldsymbol{s} \,|\, \Phi \rangle_0 = \langle \Psi_{\mathrm{T}} \,|\, \boldsymbol{s} \,|\, \Phi_{\mathrm{T}} \rangle_0 . \qquad (6.9)$$

Thus the $\boldsymbol{s}$-bracket will vanish for operators with opposite behaviour under time reversal.

C) Solution of the Linearized Transport Equation by Expansion with Respect to Irreducible Tensors

§ 7. The orthonormalized tensors

The standard distribution function in thermal equilibrium does not depend on the direction of the velocity vector $\boldsymbol{v}$ nor on the spin vector $\boldsymbol{s}$, see (4.2). But the deviation of the distribution from standard equilibrium in general depends on $\boldsymbol{v}$ and $\boldsymbol{s}$. We shall expand Φ in terms of a complete set of orthogonal functions depending on $\boldsymbol{v}$ and $\boldsymbol{s}$. The expansion coefficients will be shown to have a simple physical meaning if we choose the special orthogonal system of functions to be presented here. The procedure is analogous to that used by Waldmann in the discussion of the Lorentzian gas of particles with spin [3] and of classical rotating molecules [8].

Instead of $\boldsymbol{v}$ we use the dimensionless variable $\mathbf{V}$

$$\mathbf{V} = \sqrt{\frac{m}{2\,k\,T_0}}\,\boldsymbol{v} \tag{7.1}$$

and put

$$f_0(\boldsymbol{v})\,\mathrm{d}^3 v = \frac{n_0}{(2\,S+1)}\,\pi^{-3/2}\,e^{-V^2}\,\mathrm{d}^3 V = f_0(\mathbf{V})\,\mathrm{d}^3 V.$$

As an orthogonal system in $\mathbf{V}$-space we choose the eigensolutions of the quantum mechanical harmonic oscillator

$$\Phi_{lm}^{(n)}(\mathbf{V}) \propto S_{l+\frac{1}{2}}^{(n)}(V^2)\,\mathbf{V}^l\,Y_{lm}(\hat{\mathbf{V}}) \tag{7.2}$$

with $l, n = 0, 1, 2, \ldots$. The $S^{(n)}$ are Sonine polynomials of the n-th power in V^2. The spherical harmonics Y depend on the polar angles of $\hat{\mathbf{V}} = \mathbf{V}/|\mathbf{V}|$. The "angular momentum quantum number" $l = 0, 1, 2 \ldots$ corresponds to scalar, vector, irreducible 2nd-rank tensor etc. $N = 2\,n + l$ is the highest power occuring in $\mathbf{V}$. Instead of the above representation in polar coordinates we shall use Cartesian components. The first few nembers of this orthogonal system are listed in Tabe 1.

As an orthogonal set in $\boldsymbol{s}$-space we take the irreducible tensors which can be constructed from the Cartesian components of $\boldsymbol{s}$. The first few members are

$$1,\ s_\mu,\ \overline{s_\mu\,s_\nu} \equiv \tfrac{1}{2}(s_\mu\,s_\nu + s_\mu\,s_\nu) - \tfrac{1}{3}\,S(S+1)\,\delta_{\mu\nu}\,.$$

power N	scalars $l=0$	vectors $l=1$	irreducible 2nd rank tensors $l=2$
0	1		
1		V_μ	
2	$V^2 - \frac{3}{2}$		$V_\mu V_\nu - \frac{1}{3} V^2 \delta_{\mu\nu}$ $\equiv \overline{V_\mu V_\nu}$
3		$(V^2 - \frac{5}{2})\,V_\mu$	
4	$V^4 - 5\,V^2 + \frac{15}{4}$		$(V^2 - \frac{7}{2})\,\overline{V_\mu V_\nu}$

Table. 1. Orthogonal system in $\mathbf{V}$ -space.

As the irreducible tensor of rank $n > 2\,S$ vanishes for spin S, retaining the first two (three) terms is exact for spin $S = \frac{1}{2}$ (1).

We obtain the complete set of orthogonal functions by combining the irreducible tensors in $\mathbf{V}$- and $\boldsymbol{s}$-space. We will not handle the whole complete set but we shall truncate the pertaining infinite system by retaining only the first few members up to tensors of 2nd rank and of 4th power in $\mathbf{V}$, $\boldsymbol{s}$. Table 2 shows the first composite tensors. For the ε-tensor see [10]. In the columns P, T we note the parity and time reversal behaviour of the tensors (see 5.8; 5.12). $L = 0, 1, 2 \ldots$ characterizes scalars, vectors, 2nd rank irreducible tensors, ... True scalars ($P\,L = +0$), polar vectors ($P\,L = -1$), true 2nd rank tensors ($P\,L = +2$) are denoted by Φ, Φ_μ, $\Phi_{\mu\nu}$, pseudo scalars ($P\,L = -0$), pseudo vectors ($P\,L = +1$), pseudo tensors ($P\,L = -2$) are denoted by Ψ, Ψ_μ, $\Psi_{\mu\nu}$. Quantities of the same kind are distinguished by consecutive superscripts. We have used the following abbreviations for the irreducible Cartesian tensors (symmetric, vanishing traces)

$$\overline{a_\mu\,b_\nu} = \tfrac{1}{2}(a_\mu\,b_\nu + a_\nu\,b_\mu) - \tfrac{1}{3}\,a_\varrho\,b_\varrho\,\delta_{\mu\nu}\,, \tag{7.3}$$

$$\overline{a_\mu\,a_\nu\,a_\lambda} = a_\mu\,a_\nu\,a_\lambda - \tfrac{1}{5}\,a_\varrho\,a_\varrho(a_\mu\,\delta_{\nu\lambda} + a_\nu\,\delta_{\mu\lambda} + a_\lambda\,\delta_{\mu\nu})\,. \tag{7.4}$$

Tensors with different L's or different P's are orthogonal. The normalisation constants in Table 2 are such that the following orthonormalization relations are valid for tensors of equal rank and equal parity:

$$\text{scalars} \quad \langle \Phi^{(i)}\,\Phi^{(k)} \rangle_0 = \langle \Psi^{(i)}\,\Psi^{(k)} \rangle_0 = \delta^{(ik)}, \tag{7.5}$$

$$\text{vectors} \quad \langle \Phi_\mu^{(i)}\,\Phi_\nu^{(\mathrm{k})} \rangle_0 = \langle \Psi_\mu^{(i)}\,\Psi_\nu^{(k)} \rangle_0 = \delta^{(ik)}\,\delta_{\mu\nu}\,, \tag{7.6}$$

$$\text{2nd rank tensors} \quad \langle \Phi_{\mu\nu}^{(i)}\,\Phi_{\mu'\nu'}^{(k)} \rangle_0 = \langle \Psi_{\mu\nu}^{(i)}\,\Psi_{\mu'\nu'}^{(k)} \rangle_0 = \delta^{(ik)}\,\Delta_{\mu\nu,\,\mu'\nu'}\,. \tag{7.7}$$

Total Power	Scalars	*PLT*	Vectors	*PLT*	2nd Rank Tensors	*PLT*
0	$\Phi^{(1)}=1$	$+0+$				
1			$\Phi_\mu^{(1)}=\sqrt{2}\,V_\mu$	$-1-$		
			$\Psi_\mu^{(1)}=\frac{\sqrt{3}}{S_0}\,s_\mu$	$+1-$		
2	$\Phi^{(2)}=\sqrt{\frac{2}{3}}\left(V^2-\frac{3}{2}\right)$	$+0+$			$\Phi_{\mu\nu}^{(1)}=\sqrt{2}\,\overline{V_\mu V_\nu}$	$+2+$
	$\Psi^{(1)}=\frac{\sqrt{2}}{S_0}\,V_\varrho s_\varrho$	$-0+$	$\Phi_\mu^{(2)}=\frac{\sqrt{3}}{S_0}\,\varepsilon_{\mu\nu\lambda}V_\nu s_\nu$	$-1+$	$\Psi_{\mu\nu}^{(1)}=\frac{\sqrt{6}}{S_0}\,\overline{V_\mu s_\nu}$	$-2+$
					$\Phi_{\mu\nu}^{(2)}=\sqrt{\frac{15}{2}}\,\frac{1}{S_0 S_1}\,\overline{s_\mu s_\nu}$	$+2+$
3			$\Phi_\mu^{(3)}=\frac{2}{\sqrt{5}}\left(V^2-\frac{5}{2}\right)V_\mu$	$-1-$		
			$\Psi_\mu^{(2)}=\frac{\sqrt{2}}{S_0}\left(V^2-\frac{3}{2}\right)s_\mu$	$+1-$		
			$\Phi_\mu^{(4)}=\frac{3}{S_0 S_1}\,\overline{s_\mu s_\nu}\,V_\nu$	$-1-$	$\Psi_{\mu\nu}^{(2)}=\frac{\sqrt{10}}{S_0 S_1}\,\overline{\varepsilon_{\mu\lambda\varkappa}\,\overline{s_\nu s_\lambda}\,V_\varkappa}$	$-2-$
			$\Psi_\mu^{(3)}=\frac{3}{S_0}\sqrt{\frac{2}{5}}\,\overline{V_\mu V_\nu}\,s_\nu$	$+1-$	$\Phi_{\mu\nu}^{(3)}=\frac{2}{S_0}\,\overline{\varepsilon_{\mu\lambda\varkappa}\,\overline{V_\nu V_\lambda}\,s_\varkappa}$	$+2-$
4	$\Phi^{(3)}=\sqrt{\frac{2}{15}}\left(V^4-5\,V^2+\frac{15}{4}\right)$	$+0+$		$-1+$	$\Phi_{\mu\nu}^{(4)}=\frac{2}{\sqrt{7}}\left(V^2-\frac{7}{2}\right)\overline{V_\mu V_\nu}$	$+2+$
	$\Psi^{(2)}=\frac{2}{\sqrt{5}}\,\frac{1}{S_0}\left(V^2-\frac{5}{2}\right)V_\varrho s_\varrho$	$-0+$	$\Phi_\mu^{(5)}=\sqrt{\frac{6}{5}}\,\frac{1}{S_0}\left(V^2-\frac{5}{2}\right)\varepsilon_{\mu\nu\lambda}V_\nu s_\lambda$		$\Psi_{\mu\nu}^{(3)}=\frac{2}{S_0}\sqrt{\frac{3}{5}}\left(V^2-\frac{5}{2}\right)\overline{V_\mu s_\nu}$	$-2+$
					$\Phi_{\mu\nu}^{(5)}=\frac{\sqrt{5}}{S_0 S_1}\left(V^2-\frac{3}{2}\right)\overline{s_\mu s_\nu}$	$+2+$
	$\Phi^{(4)}=\frac{\sqrt{3}}{S_0 S_1}\,\overline{V_\mu V_\nu}\,\overline{s_\mu s_\nu}$	$+0+$	$\Psi_\mu^{(4)}=\frac{\sqrt{6}}{S_0 S_1}\,s_{\mu\nu\lambda}\overline{V_\nu V_\varrho}\,\overline{s_\varrho s_\lambda}$	$+1+$	$\Phi_{\mu\nu}^{(6)}=\frac{3\sqrt{3}}{5 S_0 S_1}\,\overline{\overline{V_\mu V_\varrho}\,\overline{s_\varrho s_\nu}}$	$+2+$
					$\Psi_{\mu\nu}^{(4)}=\frac{2}{S_0}\sqrt{\frac{5}{7}}\,\overline{V_\mu V_\nu V_\lambda}\,s_\lambda$	$-2+$
					$\Psi_{\mu\nu}^{(5)}=\frac{5}{S_0 S_1 S_2}\,\overline{s_\mu s_\nu s_\lambda}\,V_\lambda$	$-2+$

Table 2. Composite orthonormalized system in $\boldsymbol{V}, \boldsymbol{s}$-space.
Abbreviations: $S_0=\sqrt{S(S+1)}$, $S_1=\sqrt{S(S+1)-\frac{3}{4}}$, $S_2=\sqrt{S(S+1)-2}$.

The δ's are the usual KRONECKER symbols. The isotropic 4th rank tensor $\Delta_{\mu\nu,\mu'\nu'}$ is given by

$$\Delta_{\mu\nu,\mu'\nu'}=\tfrac{1}{2}\,(\delta_{\mu\mu'}\,\delta_{\nu\nu'}+\delta_{\mu\nu'}\,\delta_{\nu\mu'})-\tfrac{1}{3}\,\delta_{\mu\nu}\,\delta_{\mu'\nu'}\,. \quad (7.8)$$

Obviously one has:

$$\Delta_{\mu\nu,\mu'\nu'}=\Delta_{\mu'\nu',\mu\nu}; \quad \Delta_{\mu\mu,\mu'\nu'}=0; \quad \Delta_{\mu\nu,\mu\nu}=5. \quad (7.9)$$

The spin traces which are needed for the calculation of the normalization constants, v- and s-brackets, are given in the appendix.

In the following the functions of Table 2 will be used only up to total power 3 (with one exception).

§ 8. Expansion of the distribution matrix. The meaning of the expansion coefficients

Now we expand the relative deviation Φ from the standard equilibrium state with respect to the complete set of orthonormalized tensors introduced in § 7:

$$\begin{aligned} f&=f_0\,(1+\Phi)\\ &=f_0\,[1+\sum_k\,(a^{(k)}\,\Phi^{(k)}+b^{(k)}\,\Psi^{(k)}+a_\mu^{(k)}\,\Phi_\mu^{(k)} \quad (8.1)\\ &\qquad +b_\mu^{(k)}\,\Psi_\mu^{(k)}+a_{\mu\nu}^{(k)}\,\Phi_{\mu\nu}^{(k)}+b_{\mu\nu}^{(k)}\,\Psi_{\mu\nu}^{(k)}+\ldots)\,]. \end{aligned}$$

The expansion coefficients $a^{(k)}$, $b^{(k)}$, $a_\mu^{(k)}$, $b_\mu^{(k)}$, ... are functions of t and $\boldsymbol{x}$.

The meaning of the expansion coefficients will be seen if we multiply (8.1) by $\Phi^{(i)}$, $\Psi^{(i)}$, $\Phi_\mu^{(i)}$, $\Psi_\mu^{(i)}$, ... respectively, integrate over $\mathrm{d}^3 v$ and take the trace on spin indices. Using the mean value formula (1.5) and the orthonormality relations (7.5 – 7.7) of the expansion tensors we see that

$$\begin{aligned} n\langle \Phi^{(i)}\rangle &= n_0(\delta^{(i1)}+a^{(i)}), & n\langle \Psi^{(i)}\rangle &= n_0\, b^{(i)},\\ n\langle \Phi_\mu^{(i)}\rangle &= n_0\, a_\mu^{(i)}, & n\langle \Psi_\mu^{(i)}\rangle &= n_0\, b_\mu^{(i)},\\ n\langle \Phi_{\mu\nu}^{(i)}\rangle &= n_0\, a_{\mu\nu}^{(i)}, & n\langle \Psi_{\mu\nu}^{(i)}\rangle &= n_0\, b_{\mu\nu}^{(i)}. \end{aligned} \qquad (8.2)$$

The coefficients may be interpreted in detail by using Table 2 and going back to the original velocity $\boldsymbol{v}$ according to

$$\boldsymbol{V} = \sqrt{\tfrac{3}{2}}\, \boldsymbol{v}/v_0 \quad \text{with} \quad v_0 = \sqrt{3\,k\,T_0/m}\,. \qquad (8.3)$$

Besides, we introduce the temperature of the gas by

$$\tfrac{3}{2}\,k\,T = \langle E\rangle \quad \text{with} \quad E = \tfrac{1}{2}\,m\,v^2, \qquad (8.4)$$

the heat flux

$$\boldsymbol{q} \approx n\langle (E-\tfrac{5}{2}\,k\,T)\,\boldsymbol{v}\rangle\,, \qquad (8.5)$$

and the pressure tensor

$$p_{\mu\nu} = \langle m\, v_\mu\, v_\nu\rangle\,. \qquad (8.6)$$

Again we use the abbreviations $S_0 = \sqrt{S(S+1)}$, $S_1 = \sqrt{S(S+1)-\tfrac{3}{4}}$. Now we are ready to give the physical meaning of the first few of the expansion coefficients:

a) scalars (8.7)

$$a^{(1)} = \frac{n-n_0}{n_0} \text{ (deviation of particle density)},$$

$$a^{(2)} = \sqrt{\frac{3}{2}}\,\frac{n}{n_0}\,\frac{T-T_0}{T_0} \text{ (deviation of temperature)}.$$

b) pseudo scalar (8.8)

$$b^{(1)} = \frac{\sqrt{3}}{S_0}\,\frac{n}{n_0\,v_0}\,\langle \boldsymbol{v}\cdot\boldsymbol{s}\rangle \text{ ("longitudinal spin")}.$$

c) polar vectors (8.9)

$$\boldsymbol{a}^{(1)} = \sqrt{3}\,\frac{n}{n_0\,v_0}\,\langle \boldsymbol{v}\rangle \quad \text{(gas velocity)},$$

$$\boldsymbol{a}^{(2)} = \frac{3}{\sqrt{2}}\,\frac{n}{S_0\,n_0\,v_0}\,\langle \boldsymbol{v}\times\boldsymbol{s}\rangle \text{ ("azimuthal spin")},$$

$$\begin{aligned}\boldsymbol{a}^{(3)} &= \sqrt{\frac{6}{5}}\,\frac{n}{n_0}\,\frac{1}{k\,T_0\,v_0}\,\langle (E-\tfrac{5}{2}\,k\,T_0)\,\boldsymbol{v}\rangle\\ &= \sqrt{\frac{6}{5}}\,\frac{1}{n_0\,k\,T_0\,v_0}\,[\boldsymbol{q}+\tfrac{5}{2}\,k(T-T_0)\,n\langle \boldsymbol{v}\rangle]\\ &\hspace{6em}\text{(energy flux)},\end{aligned}$$

$$\begin{aligned}\boldsymbol{a}^{(4)} &= 3\sqrt{\frac{3}{2}}\,\frac{1}{S_0\,S_1}\,\frac{n}{n_0\,v_0}\\ &\hspace{4em}\langle \tfrac{1}{2}\,(\boldsymbol{s}\cdot\boldsymbol{v}\;\boldsymbol{s}+\boldsymbol{s}\;\boldsymbol{v}\cdot\boldsymbol{s}) - \tfrac{1}{3}\,S_0^{\,2}\,\boldsymbol{v}\rangle\,,\end{aligned}$$

$$\begin{aligned}\boldsymbol{a}^{(5)} &= \frac{3}{\sqrt{5}}\,\frac{1}{S_0}\,\frac{n}{n_0\,k\,T_0\,v_0}\,\langle (E-\tfrac{5}{2}\,k\,T_0)\,\boldsymbol{v}\times\boldsymbol{s}\rangle\\ &= \frac{3}{\sqrt{5}}\,\frac{n}{n_0\,S_0\,k\,T_0\,v_0}\,\langle E\,\boldsymbol{v}\times\boldsymbol{s}\rangle - \sqrt{\frac{5}{2}}\,\boldsymbol{a}^{(2)}.\end{aligned}$$

d) pseudo vectors (8.10)

$$\boldsymbol{b}^{(1)} = \frac{\sqrt{3}}{S_0}\,\frac{n}{n_0}\,\langle \boldsymbol{s}\rangle \quad \text{(vector polarization)},$$

$$\begin{aligned}\boldsymbol{b}^{(2)} &= \frac{3}{\sqrt{2}}\,\frac{n}{S_0\,n_0\,E_0}\,\langle (E-E_0)\,\boldsymbol{s}\rangle\\ &= \frac{3}{\sqrt{2}}\,\frac{n}{S_0\,n_0\,v_0^{\,2}}\,\langle v^2\,\boldsymbol{s}\rangle - \sqrt{\frac{3}{2}}\,\boldsymbol{b}^{(1)},\end{aligned}$$

$$\boldsymbol{b}^{(3)} = \frac{9}{\sqrt{10}}\,\frac{n}{S_0\,n_0\,v_0^{\,2}}\,\langle \boldsymbol{v}\;\boldsymbol{v}\cdot\boldsymbol{s} - \tfrac{1}{3}\,v^2\,\boldsymbol{s}\rangle\,.$$

e) 2nd rank tensors 8.11)

$$a_{\mu\nu}^{(1)} = \frac{3}{\sqrt{2}}\,\frac{n}{n_0\,v_0^{\,2}}\,\langle v_\mu\,v_\nu - \tfrac{1}{3}v^2\,\delta_{\mu\nu}\rangle \quad \text{(friction tensor)},$$

$$a_{\mu\nu}^{(2)} = \sqrt{\frac{15}{2}}\,\frac{n}{S_0\,S_1\,n_0}\,\langle \overline{s_\mu\,s_\nu}\rangle \quad \text{(tensor polarization)},$$

$$a_{\mu\nu}^{(3)} = \frac{3}{S_0}\,\frac{n}{n_0\,v_0^{\,2}}\,\langle \overline{\varepsilon_{\mu\lambda\varkappa}\,\overline{v_\nu\,v_\lambda}\,s_\varkappa}\rangle\,.$$

f) 2nd rank pseudo tensors (8.12)

$$b_{\mu\nu}^{(1)} = \frac{3}{S_0\,v_0}\,\frac{n}{n_0}\,\langle \overline{v_\mu\,s_\nu}\rangle\,,$$

$$b_{\mu\nu}^{(2)} = \frac{\sqrt{15}}{S_0\,S_1}\,\frac{n}{n_0\,v_0}\,\langle \overline{\varepsilon_{\mu\lambda\varkappa}\,\overline{s_\nu\,s_\lambda}\,v_\varkappa}\rangle\,.$$

§ 9. Expansion of the transport equation

Now, we insert the expansion (8.1) for Φ into the linearized BOLTZMANN equation (4.6). Multiplying by $f_0\,\Phi^{(i)}$, $f_0\,\Psi^{(i)}$, $f_0\,\Phi_\mu^{(i)}$, $f_0\,\Psi_\mu^{(i)}$, ... respectively, integrating over $\mathrm{d}^3 v_1$, and applying the trace tr_1, we obtain a set of coupled linear differential equations for the expansion coefficients $a^{(i)}$, $b^{(i)}$, $a_\mu^{(i)}$, $b_\mu^{(i)}$, ...

We shall discuss the different terms of these equations separately.

a) $\partial/\partial t$-term

Due to the orthonormalization of the expansion tensors we simply obtain from the $\partial\Phi/\partial t$ term in the BOLTZMANN equation

$$\frac{\partial a^{(i)}}{\partial t}\,,\quad \frac{\partial b^{(i)}}{\partial t}\,,\quad \frac{\partial a_\mu^{(i)}}{\partial t}\,,\quad \frac{\partial b_\mu^{(i)}}{\partial t}\,,\;\dots\,.$$

b) Convection term

The convection term $v_\mu(\partial\Phi/\partial x_\mu)$ leads to the $\boldsymbol{v}$-brackets defined in § 6. The $\boldsymbol{v}$-brackets vanish for expansion tensors of equal parities (see § 6). Further general relations follow from the isotropy of space, e. g.

$$\langle \Phi^{(i)}\,|\,v_\mu\,|\,\Psi^{(k)}\rangle_0 = 0\,,$$

$$\langle \Phi^{(i)}\,|\,v_\mu\,|\,\Psi_{\nu\lambda}^{(k)}\rangle_0 = 0\,;$$

these quantities must vanish due to the absence of a preferential direction. Introducing "reduced" brackets

$\langle P\,L\,i\,|\,v\,|-P\,L'\,k\rangle$, we now list the $\boldsymbol{v}$-brackets which may be non-zero from these general reasons.

$$\begin{aligned}\langle \Phi^{(i)}\,|\,v_\nu\,|\,\Phi^{(k)}_\lambda\rangle_0&=\langle +0\,i\,|\,v\,|\,-1\,k\rangle\,\delta_{\nu\lambda}\,,\\ \langle \Psi^{(i)}\,|\,v_\nu\,|\,\Psi^{(k)}_\lambda\rangle_0&=\langle -0\,i\,|\,v\,|\,+1\,k\rangle\,\delta_{\nu\lambda}\,,\end{aligned}\tag{9.1}$$

$$\langle \Phi^{(i)}_\mu\,|\,v_\nu\,|\,\Psi^{(k)}_\lambda\rangle_0=\langle -1\,i\,|\,v\,|\,+1\,k\rangle\,\varepsilon_{\mu\nu\lambda}\,,\tag{9.2}$$

$$\begin{aligned}\langle \Phi^{(i)}_\mu\,|\,v_\nu\,|\,\Phi^{(k)}_{\mu'\nu'}\rangle_0&=\langle -1\,i\,|\,v\,|\,+2\,k\rangle\,\triangle_{\mu\nu,\,\mu'\nu'}\,,\\ \langle \Psi^{(i)}_\mu\,|\,v_\nu\,|\,\Psi^{(k)}_{\mu'\nu'}\rangle_0&=\langle +1\,i\,|\,v\,|\,-2\,k\rangle\,\triangle_{\mu\nu,\,\mu'\nu'}\,,\end{aligned}\tag{9.3}$$

$$\langle \Phi^{(i)}_{\mu\nu}\,|\,v_\lambda\,|\,\Psi^{(k)}_{\mu'\nu'}\rangle_0=\langle +2\,i\,|\,v\,|\,-2\,k\rangle\,\Box_{\mu\nu,\,\lambda,\,\mu'\nu'}\,.\tag{9.4}$$

This list has to be completed by the symmetry (6.3) valid for any two function Φ and Ψ (no distinction had been made in § 6 between proper and pseudo quantities). The isotropic 4th rank tensor $\triangle$ has been defined in (7.8). The new one $\Box_{\mu\nu,\,\lambda,\,\mu'\nu'}$ is an isotropic tensor of 5th rank, namely

$$\begin{aligned}&\Box_{\mu\nu,\,\lambda,\,\mu'\nu'}\\ &=\tfrac{1}{4}\,(\delta_{\mu\mu'}\,\varepsilon_{\nu\lambda\nu'}+\delta_{\mu\nu'}\,\varepsilon_{\nu\lambda\mu'}+\delta_{\nu\mu'}\,\varepsilon_{\mu\lambda\nu'}+\delta_{\nu\nu'}\,\varepsilon_{\mu\lambda\mu'})\,.\end{aligned}\tag{9.5}$$

It has the properties:

$$\begin{aligned}&\Box_{\mu\nu,\,\lambda,\,\mu'\nu'}=-\Box_{\mu'\nu',\,\lambda,\,\mu\nu}\,, &&\Box_{\mu\mu,\,\lambda,\,\mu'\nu'}=0\,,\\ &\Box_{\mu\nu,\,\lambda,\,\mu\nu'}=\tfrac{5}{4}\,\varepsilon_{\nu\lambda\nu'}\,, &&\Box_{\mu\nu,\,\lambda,\,\mu\nu'}\,\varepsilon_{\nu\lambda\nu'}=\tfrac{15}{2}\,.\end{aligned}\tag{9.6}$$

We obtain the reduced brackets themselves by contracting and noting

$$\delta_{\mu\mu}=3,\quad \varepsilon_{\mu\nu\lambda}\,\varepsilon_{\mu\nu\lambda}=6,\quad \triangle_{\mu\nu,\,\mu\nu}=5.$$

This gives

$$\begin{aligned}\langle +0\,i\,|\,v\,|\,-1\,k\rangle&=\tfrac{1}{3}\,\langle \Phi^{(i)}\,v_\varrho\,\Phi^{(k)}_\varrho\rangle_0\,,\\ \langle -0\,i\,|\,v\,|\,+1\,k\rangle&=\tfrac{1}{3}\,\langle \Psi^{(i)}\,v_\varrho\,\Psi^{(k)}_\varrho\rangle_0\,,\\ \langle -1\,i\,|\,v\,|\,+1\,k\rangle&=\tfrac{1}{6}\,\varepsilon_{\mu\nu\lambda}\langle \Phi^{(i)}_\mu\,v_\nu\,\Psi^{(k)}_\lambda\rangle_0\\ &=\tfrac{1}{6}\,\langle\,(\boldsymbol{\Phi}^{(i)}\times\boldsymbol{v})\cdot\boldsymbol{\Psi}^{(k)}\rangle_0\,,\\ \langle -1\,i\,|\,v\,|\,+2\,k\rangle&=\tfrac{1}{5}\,\langle \Phi^{(i)}_\mu\,v_\nu\,\Phi^{(k)}_{\mu\nu}\rangle_0\,,\\ \langle +1\,i\,|\,v\,|\,-2\,k\rangle&=\tfrac{1}{5}\,\langle \Psi^{(i)}_\mu\,v_\nu\,\Psi^{(k)}_{\mu\nu}\rangle_0\,,\\ \langle +2\,i\,|\,v\,|\,-2\,k\rangle&=\tfrac{2}{15}\,\varepsilon_{\nu\lambda\nu'}\langle \Phi^{(i)}_{\mu\nu}\,v_\lambda\,\Psi^{(k)}_{\mu\nu'}\rangle_0\,.\end{aligned}\tag{9.7}$$

Keeping in mind that

$$\langle\ldots|\,v\,|\ldots\rangle=\sqrt{\tfrac{2}{3}}\;v_0\langle\ldots|\,V\,|\ldots\rangle$$

we may evaluate the reduced matrix elements by inserting the functions from Table 2 and performing the trace and the V-integration. The results are incorporated in the expansion equations given in § 11.

c) Spin precession term

With a homogeneous magnetic field the transport equation contains the term $-\,i\,\omega_{\mathrm{H}}(\boldsymbol{h}\cdot\boldsymbol{s}\,f-f\,\boldsymbol{h}\cdot\boldsymbol{s})$. This term leads to the $\boldsymbol{s}$-brackets defined by (6.2). As shown in § 6, the $\boldsymbol{s}$-brackets vanish for two tensors with different P- and T-behaviour. Due to this fact and due to the isotropy of space only the following $\boldsymbol{s}$-brackets occur:

$$\begin{aligned}\langle \Phi^{(\mathrm{i})}\,|\,s_\nu\,|\,\Psi^{(k)}_\lambda\rangle_0&=\langle +0\,i\,|\,s\,|\,+1\,k\rangle\,\delta_{\nu\lambda}\,,\\ \langle \Psi^{(i)}\,|\,s_\nu\,|\,\Phi^{(k)}_\lambda\rangle_0&=\langle -0\,i\,|\,s\,|\,-1k\rangle\,\delta_{\nu\lambda}\,,\end{aligned}\tag{9.8}$$

$$\begin{aligned}\langle \Phi^{(i)}_\mu|\,s_\nu\,|\,\Phi^{(k)}_\lambda\rangle_0&=\langle -1\,i\,|\,s\,|\,-1\,k\rangle\,\varepsilon_{\mu\nu\lambda}\,,\\ \langle \Psi^{(i)}_\mu|\,s_\nu\,|\,\Psi^{(k)}_\lambda\rangle_0&=\langle +1\,i\,|\,s\,|\,+1\,k\rangle\,\varepsilon_{\mu\nu\lambda}\,,\end{aligned}\tag{9.9}$$

$$\begin{aligned}\langle \Phi^{(i)}_\mu|\,s_\nu\,|\,\Psi^{(k)}_{\mu'\nu'}\rangle_0&=\langle -1\,i\,|\,s\,|\,-2\,k\rangle\,\triangle_{\mu\nu,\,\mu'\nu'}\,,\\ \langle \Psi^{(i)}_\mu|\,s_\nu\,|\,\Phi^{(k)}_{\mu'\nu'}\rangle_0&=\langle +1\,i\,|\,s\,|\,+2\,k\rangle\triangle_{\mu\nu,\,\mu'\nu'}\,,\end{aligned}\tag{9.10}$$

$$\begin{aligned}\langle \Phi^{(i)}_{\mu\nu}|\,s_\lambda\,|\,\Phi^{(k)}_{\mu'\nu'}\rangle_0&=\langle +2\,i\,|\,s\,|\,+2\,k\rangle\,\Box_{\mu\nu,\,\lambda,\,\mu'\nu'}\,,\\ \langle \Psi^{(i)}_{\mu\nu}|\,s_\lambda\,|\,\Psi^{(k)}_{\mu'\nu'}\rangle_0&=\langle -2\,i\,|\,s\,|\,-2\,k\rangle\,\Box_{\mu\nu,\,\lambda,\,\mu'\nu'}\,.\end{aligned}\tag{9.11}$$

The list has to be completed by the antisymmetry (6.7). The reduced s-brackets $\langle P\,L\,i\,|\,s\,|\,P\,\mathrm{L}'\,k\rangle$ may befound by contraction (similar to the v-brackets).

d) Collision term

The collision operator ω maps a tensor of given rank and parity into a tensor of the same rank and parity. Due to this fact and to the isotropy of space in general only the following non-zero collision brackets occur:

$$\begin{aligned}\langle \Phi^{(i)}\,\omega(\Phi^{(k)})\,\rangle_0&=\langle +0\,i\,|\,\omega\,|\,+0\,k\rangle\equiv\omega^{(ik)}_{+0}\,,\\ \langle \Psi^{(i)}\,\omega(\Psi^{(k)})\,\rangle_0&=\langle -0\,i\,|\,\omega\,|\,-0\,k\rangle\equiv\omega^{(ik)}_{-0}\,,\end{aligned}\tag{9.12}$$

$$\begin{aligned}\langle \Phi^{(i)}_\mu\omega(\Phi^{(k)}_\nu)\,\rangle_0&=\langle -1\,i\,|\,\omega\,|-1\,k\rangle\,\delta_{\mu\nu}\equiv\omega^{(ik)}_{-1}\,\delta_{\mu\nu}\,,\\ \langle \Psi^{(i)}_\mu\omega(\Psi^{(k)}_\nu)\,\rangle_0&=\langle +1\,i\,|\,\omega\,|+1\,k\rangle\,\delta_{\mu\nu}\equiv\omega^{(ik)}_{+1}\,\delta_{\mu\nu}\,,\end{aligned}\tag{9.13}$$

$$\begin{aligned}\langle \Phi^{(i)}_{\mu\nu}\omega(\Phi^{(k)}_{\mu'\nu'})\,\rangle_0&=\langle +2\,i\,|\,\omega\,|+2\,k\rangle\,\triangle_{\mu\nu,\,\mu'\nu'}\\ &\equiv\omega^{(ik)}_{+2}\,\triangle_{\mu\nu,\mu'\nu'}\,,\\ \langle \Psi^{(i)}_{\mu\nu}\omega(\Psi^{(k)}_{\mu'\nu'})\,\rangle_0&=\langle -2\,i\,|\,\omega\,|-2\,k\rangle\,\triangle_{\mu\nu,\,\mu'\nu'}\\ &\equiv\omega^{(ik)}_{-2}\,\triangledown_{\mu\nu,\mu'\nu'}\,.\end{aligned}\tag{9.14}$$

The reduced collision brackets $\langle P\,l\,i\,|\,\omega\,|\,P\,L\,k\rangle$ for which a second, simpler notation has been introduced, may be found by contraction:

$$\begin{aligned}\omega^{(ik)}_{-1}&=\tfrac{1}{3}\,\langle \Phi^{(i)}_\varrho\,\omega\,(\Phi^{(k)}_\varrho)\,\rangle_0\,,\\ \omega^{(ik)}_{+1}&=\tfrac{1}{3}\,\langle \Psi^{(i)}_\varrho\,\omega\,(\Psi^{(k)}_\varrho)\,\rangle_0\,,\\ \omega^{(ik)}_{+2}&=\tfrac{1}{5}\,\langle \Phi^{(i)}_{\varrho\varkappa}\,\omega\,(\Phi^{(k)}_{\varrho\varkappa})\,\rangle_0\,,\\ \omega^{(ik)}_{-2}&=\tfrac{1}{5}\,\langle \Psi^{(i)}_{\varrho\varkappa}\,\omega\,(\Psi^{(k)}_{\varrho\varkappa})\,\rangle_0\,.\end{aligned}\tag{9.15}$$

These coefficients are termed relaxation coefficients of the system.

By our expansion procedure the transport equation which is an integro-different equation for the distribution operator is reduced to a system of coupled linear differential equations for the expansion coefficients of the distribution operator. These

$$\frac{\partial a^{(i)}}{\partial t}+\sum_k\left[\langle +0\,i\,|\,v\,|\,-1\,k\rangle\,\operatorname{div}\boldsymbol{a}^{(k)}-i\,\omega_{\mathrm H}\langle +0\,i\,|\,s\,|\,+1\,k\rangle\;\boldsymbol{h}\cdot\boldsymbol{b}^{(k)}+\omega^{(ik)}_{+0}\,a^{(k)}\right]=0$$

$$\frac{\partial b^{(i)}}{\partial t}+\sum_k\left[-i\,\omega_{\mathrm H}\langle -0\,i\,|\,s\,|\,-1\,k\rangle\;\boldsymbol{h}\cdot\boldsymbol{a}^{(k)}+\langle -0\,i\,|\,v\,|\,+1\,k\rangle\,\operatorname{div}\boldsymbol{b}^{(k)}+\omega^{(ik)}_{-0}\,b^{(k)}\right]=0$$

$$\frac{\partial a^{(i)}_{\mu}}{\partial t}+\sum_k\Big[\langle +0\,k\,|\,v\,|\,-1\,i\rangle\,\operatorname{grad}_{\mu}a^{(k)}+i\,\omega_{\mathrm H}\langle -0\,k\,|\,s\,|\,-1\,i\rangle\,h_{\mu}\,b^{(k)}-i\,\omega_{\mathrm H}\langle -1\,i\,|\,s\,|\,-1\,k\rangle\,(\boldsymbol{h}\times\boldsymbol{a}^{(k)})_{\mu}$$
$$+\langle -1\,i\,|\,v\,|\,+1\,k\rangle\,\operatorname{rot}_{\mu}\boldsymbol{b}^{(k)}+\langle -1\,i\,|\,v\,|\,+2\,k\rangle\,\frac{\partial a^{(k)}_{\mu\nu}}{\partial x_{\nu}}-i\,\omega_{\mathrm H}\langle -1\,i\,|\,s\,|\,-2\,k\rangle\,h_{\nu}\,b^{(k)}_{\mu\nu}+\omega^{(ik)}_{-1}\,a^{(k)}_{\mu}\Big]=0$$

$$\frac{\partial b^{(i)}_{\mu}}{\partial t}+\sum_k\Big[i\,\omega_{\mathrm H}\langle +0\,k\,|\,s\,|\,+1\,i\rangle\,h_{\mu}\,a^{(k)}+\langle -0\,k\,|\,v\,|\,+1\,i\rangle\,\operatorname{grad}_{\mu}b^{(k)}-\langle -1\,k\,|\,v\,|\,+1\,i\rangle\,\operatorname{rot}_{\mu}\boldsymbol{a}^{(k)}$$
$$-i\,\omega_{\mathrm H}\langle +1\,i\,|\,s\,|\,+1\,k\rangle\,(\boldsymbol{h}\times\boldsymbol{b}^{(k)})_{\mu}-i\,\omega_{\mathrm H}\langle +1\,i\,|\,s\,|\,+2\,k\rangle\,h_{\nu}\,a^{(k)}_{\mu\nu}+\langle +1\,i\,|\,v\,|\,-2\,k\rangle\,\frac{\partial b^{(k)}_{\mu\nu}}{\partial x_{\nu}}+\omega^{(ik)}_{+1}\,b^{(k)}_{\mu}\Big]=0$$

$$\frac{\partial a^{(i)}_{\mu\nu}}{\partial t}+\sum_k\Big[\langle -1\,k\,|\,v\,|\,+2\,i\rangle\,\overline{\frac{\partial a^{(k)}_{\mu}}{\partial x_{\nu}}}+i\,\omega_{\mathrm H}\langle +1\,k\,|\,s\,|\,+2\,i\rangle\,\overline{h_{\mu}\,b^{(k)}_{\nu}}-i\,\omega_{\mathrm H}\langle +2\,i\,|\,s\,|\,+2\,k\rangle\,\overline{\varepsilon_{\mu\varkappa\lambda}\,h_{\varkappa}\,a^{(k)}_{\lambda\nu}}$$
$$+\langle +2\,i\,|\,v\,|\,-2\,k\rangle\,\overline{\varepsilon_{\mu\varkappa\lambda}\frac{\partial b^{(k)}_{-\lambda\nu}}{\partial x_{\varkappa}}}\qquad+\omega^{(ik)}_{+2}\,a^{(k)}_{\mu\nu}\Big]=0$$

$$\frac{\partial b^{(i)}_{\mu\nu}}{\partial t}+\sum_k\Big[i\,\omega_{\mathrm H}\langle -1\,k\,|\,s\,|\,-2\,i\rangle\,\overline{h_{\mu}a^{(k)}_{\nu}}+\langle +1\,k\,|\,v\,|\,-2\,i\rangle\,\overline{\frac{\partial b^{(k)}_{\mu}}{\partial x_{\nu}}}-\langle +2\,k\,|\,v\,|\,-2\,i\rangle\,\overline{\varepsilon_{\mu\varkappa\lambda}\,\frac{\partial a^{(i)}_{\lambda\nu}}{\partial x_{\varkappa}}}$$
$$-i\,\omega_{\mathrm H}\langle -2\,i\,|\,s\,|\,-2\,k\rangle\,\overline{\varepsilon_{\mu\varkappa\lambda}\,h_{\varkappa}\,b^{(k)}_{\lambda\nu}}+\omega^{(ik)}_{-2}\,b^{(k)}_{\mu\nu}\Big]=0$$

Table 3. The General System of Transport-Relaxation Equations.

coupled differential equations are called the "transport-relaxation-equations". They are listed in Table 3. Before stating them more explicitly, we shall recall the general relations for the relaxation coefficients resulting from the time reversal invariance and the conservation laws.

§ 10. *Onsager-Casimir symmetry relations for the relaxation coefficients. Conservation laws*

Indicating the T-behaviour of the expansion tensors explicitly the general non-zero relaxation coefficient reads $\langle P\,L\,T\,k\,|\,\omega\,|\,P\,L\,T'\,k'\rangle$. According to (5.16) one has

$$\langle P\,L\,T\,k\,|\,\omega\,|\,P\,L\,T'\,k'\rangle=T'\,T\langle P\,L\,T'\,k'\,|\,\omega\,|\,P\,L\,T\,k\rangle. \tag{10.1}$$

This is the general ONSAGER-CASIMIR relation. As has been shown in § 5, it follows from the microreversibility of the binary collision.

Thus according to Table 2 we have the following symmetries of the first few relaxation coefficients:

$$\omega^{(ik)}_{+0}=\omega^{(ki)}_{+0}\ \text{for}\ i,k=1,2,3,4;$$
$$\omega^{(ik)}_{-0}=\omega^{(ki)}_{-0}\ \text{for}\ i,k=1,2; \tag{10.2}$$

$$\omega^{(ik)}_{-1}=\omega^{(ki)}_{-1}\ \text{for}\ i,k=1,3,4\ \text{and}\ i,k=2,5;$$
$$\omega^{(ik)}_{-1}=-\omega^{(ik)}_{-1}\text{for}\ i=2,5;\ k=1,3,4; \tag{10.3}$$

$$\omega^{(ik)}_{+1}=\omega^{(ik)}_{+1}\ \text{for}\ i,k=1,2,3\ \text{and}\ i=k=4;$$
$$\omega^{(ik)}_{+1}=-\omega^{(ki)}_{+1}\ \text{for}\ i=1,2,3;\ k=4; \tag{10.4}$$

$$\omega^{(ik)}_{+2}=\omega^{(ki)}_{+2}\ \text{for}\ i,k=1,2,4,5,6\ \text{and}\ i=k=3;$$
$$\omega^{(ik)}_{+2}=-\omega^{(ki)}_{+2}\ \text{for}\ i=1,2,4,5,6;\ k=3; \tag{10.5}$$

$$\omega^{(ik)}_{-2}=\omega^{(ki)}_{-2}\ \text{for}\ i,k=1,3,4,5\ \text{and}\ i=k=2;$$
$$\omega^{(ik)}_{-2}=-\omega^{(ki)}_{-2}\ \text{for}\ i=1,3,4,5;\ k=2. \tag{10.6}$$

Due to the above symmetries and due to the conservation laws (4.11) for the number of particles, the energy and the momentum one has respectively:

$$\begin{aligned}\omega^{(i1)}_{+0}&=\omega^{(1i)}_{+0}=0,\\ \omega^{(i2)}_{+0}&=\omega^{(2i)}_{+0}=0,\\ \omega^{(i1)}_{-1}&=\omega^{(1i)}_{-1}=0.\end{aligned} \tag{10.7}$$

§ 11. *Explicit truncated form of the transport-relaxation-equations*

We now truncate the infinite system of linear differential equations for the expansion coefficients which is equivalent to the linearized BOLTZMANN equation. We retain only those coefficients pertaining to the expansion tensors up to "total power" 3 and 2nd rank of Table 2. But in addition to this, one coefficient pertaining to "total power" 4 has been incorporated (namely $\boldsymbol{a}^{(5)}$); it will be needed in the discussion. After evaluating the required v- and s-brackets and using (10.7), one obtains the following finite system of transport-relaxation-equations [see (8.3) for v_0; $\boldsymbol{h}$ is recalled as the unit vector of direction of the magnetic field]:

a) *Scalars* (11.1)

$$\frac{\partial a^{(1)}}{\partial t}+\frac{v_0}{3}\sqrt{3}\,\mathrm{div}\,\boldsymbol{a}^{(1)}=0,$$

$$\frac{\partial a^{(2)}}{\partial t}+\frac{v_0}{3}\mathrm{div}(\sqrt{2}\,\boldsymbol{a}^{(1)}+\sqrt{5}\,\boldsymbol{a}^{(3)})=0.$$

b) *Pseudo scalars* (11.2)

$$\frac{\partial b^{(1)}}{\partial t}+\frac{v_0}{3}\mathrm{div}\,\boldsymbol{b}^{(1)}+\sqrt{\tfrac{2}{3}}\,\boldsymbol{b}^{(2)}+\sqrt{\tfrac{10}{3}}\,\boldsymbol{b}^{(3)})-\sqrt{\tfrac{2}{3}}\,\omega_h\,\boldsymbol{h}\cdot\boldsymbol{a}^{(2)}\qquad+\omega_{-0}^{(11)}\,b^{(1)}=0.$$

c) *Vectors* (11.3)

$$\frac{\partial a_\mu^{(1)}}{\partial t}+\frac{v_0}{3}\mathrm{grad}_\mu(\sqrt{3}\,a^{(1)}+\sqrt{2}\,a^{(2)})\;+v_0\sqrt{\frac{2}{3}}\frac{\partial a_{\mu\nu}^{(1)}}{\partial x_\nu}=0,$$

$$\frac{\partial a_\mu^{(2)}}{\partial t}+\sqrt{\frac{2}{3}}\,\omega_H\,h_\mu\,b^{(1)}+\frac{1}{2}\,\omega_H(\boldsymbol{h}\times\boldsymbol{a}^{(2)})_\mu+\frac{v_0}{6}\mathrm{rot}_\mu(\sqrt{6}\,\boldsymbol{b}^{(1)}+2\,\boldsymbol{b}^{(2)}+\sqrt{5}\,\boldsymbol{b}^{(3)})+$$
$$+\frac{2}{3}\,v_0\sqrt{2}\,\frac{\partial a_{\mu\nu}^{(3)}}{\partial x_\nu}-\frac{1}{\sqrt{2}}\,\omega_H\,h_\nu\,b_{\mu\nu}^{(1)}\qquad+\sum_{k=2}^{5}\omega_{-1}^{(2k)}\,a_\mu^{(k)}=0,$$

$$\frac{\partial a_\mu^{(3)}}{\partial t}+\frac{v_0}{3}\mathrm{grad}_\mu\sqrt{5}\,a^{(2)}+\frac{v_0}{3}\,2\sqrt{\frac{3}{5}}\frac{\partial a_{\mu\nu}^{(1)}}{\partial x_\nu}\qquad+\sum_{k=2}^{5}\omega_{-1}^{(3k)}\,a_\mu^{(k)}=0,$$

$$\frac{\partial a_\mu^{(4)}}{\partial t}+\frac{3}{2}\,\omega_H(\boldsymbol{h}\times\boldsymbol{a}^{(4)})_\mu+\frac{v_0}{5}\sqrt{5}\,\frac{\partial a_{\mu\nu}^{(2)}}{\partial x_\nu}+\frac{3}{\sqrt{10}}\,\omega_{\mathrm{H}}\,h_\nu\,b_{\mu\nu}^{(2)}\qquad+\sum_{k=2}^{5}\omega_{-1}^{(4k)}\,a_\mu=0,$$

$$\frac{\partial a_\mu^{(5)}}{\partial t}+\frac{1}{2}\,\omega_H(\boldsymbol{h}\times\boldsymbol{a}^{(5)})_\mu+\frac{v_0}{6}\mathrm{rot}_\mu(\sqrt{10}\,\boldsymbol{b}^{(2)}-\sqrt{2}\,\boldsymbol{b}^{(3)})+\frac{v_0}{5}\sqrt{5}\,\frac{\partial a_{\mu\nu}^{(3)}}{\partial x_\nu}\qquad+\sum_{k=2}^{5}\omega_{-1}^{(5k)}\,a_\mu^{(k)}=0.$$

d) *Pseudo vectors* (11.4)

$$\frac{\partial b_\mu^{(1)}}{\partial t}+\frac{v_0}{3}\mathrm{grad}_\mu\,b^{(1)}-\frac{v_0}{\sqrt{6}}\mathrm{rot}_\mu\,\boldsymbol{a}^{(2)}+\omega_H(\boldsymbol{h}\times\boldsymbol{b}^{(1)})_\mu+\frac{v_0}{\sqrt{3}}\frac{\partial b_{\mu\nu}^{(1)}}{\partial x_\nu}\qquad+\sum_{k=1}^{3}\omega_{+1}^{(1k)}\,b_\mu^{(k)}=0,$$

$$\frac{\partial b_\mu^{(2)}}{\partial t}+\frac{v_0}{3}\mathrm{grad}_\mu\left(\sqrt{\frac{2}{3}}\,b^{(1)}+4\sqrt{\frac{5}{3}}\,b^{(2)}\right)-\frac{v_0}{6}\mathrm{rot}_\mu(2\,\boldsymbol{a}^{(2)}+\sqrt{10}\,\boldsymbol{a}^{(5)})$$
$$+\omega_H(\boldsymbol{h}\times\boldsymbol{b}^{(2)})_\mu+\frac{v_0\sqrt{2}}{3}\frac{\partial b_{\mu\nu}^{(1)}}{\partial x_\nu}\qquad+\sum_{k=1}^{3}\omega_{+1}^{(2k)}\,b_\mu^{(k)}=0,$$

$$\frac{\partial b_\mu^{(3)}}{\partial t}+\frac{v_0}{3}\mathrm{grad}_\mu\left(\sqrt{\frac{10}{3}}\,b^{(1)}+\frac{2}{\sqrt{3}}\,b^{(2)}\right)-\frac{v_0}{6}\mathrm{rot}_\mu(\sqrt{5}\,\boldsymbol{a}^{(2)}-\sqrt{2}\,\boldsymbol{a}^{(5)})-\frac{\omega_{\mathrm{H}}}{6}(\boldsymbol{h}\times\boldsymbol{b}^{(3)})_\mu$$
$$-\omega_H\frac{3}{5}\sqrt{\frac{5}{2}}\,h_\nu\,a_{\mu\nu}^{(3)}+\frac{v_0}{6}\sqrt{\frac{2}{5}}\frac{\partial b_{\mu\nu}^{(1)}}{\partial x_\nu}\qquad+\sum_{k=1}^{3}\omega_{+1}^{(3k)}\,b_\mu^{(k)}=0.$$

e) *2nd rank tensors* (11.5)

$$\frac{\partial a_{\mu\nu}^{(1)}}{\partial t}+v_0\sqrt{\frac{2}{3}}\,\overline{\frac{\partial a_\nu^{(1)}}{\partial x_u}}+\frac{2}{3}\,v_0\sqrt{\frac{3}{5}}\,\overline{\frac{\partial a_\nu^{(3)}}{\partial x_\mu}}\qquad+\sum_{k=1}^{3}\omega_{+2}^{(1k)}\,a_{\mu\nu}^{(k)}=0,$$

$$\frac{\partial a_{\mu\nu}^{(2)}}{\partial t}+\frac{v_0}{5}\sqrt{5}\,\overline{\frac{\partial a_\nu^{(4)}}{\partial x_u}}+2\,\omega_H\,\square_{\mu\nu,\lambda,\mu'\nu'}\,h_\lambda\,a_{\mu'\nu'}^{(2)}+\frac{v_0}{3}\sqrt{2}\,\square_{\mu\nu,\lambda,\mu'\nu'}\,\frac{\partial b_{\mu'\nu'}^{(2)}}{\partial x_\lambda}\qquad+\sum_{k=1}^{3}\omega_{+2}^{(2k)}\,a_{\mu\nu}^{(k)}=0,$$

$$\frac{\partial a_{\mu\nu}^{(3)}}{\partial t}+\frac{v_0}{5}\left(\frac{10\sqrt{2}}{3}\,\overline{\frac{\partial a_\nu^{(2)}}{\partial x_u}}+\sqrt{5}\,\overline{\frac{\partial a_\nu^{(5)}}{\partial x_u}}\right)+\frac{3}{5}\,\omega_H\sqrt{\frac{5}{2}}\,\overline{h_\mu\,b_\nu^{(3)}}+\frac{3}{2}\,\omega_H\,\square_{\mu\nu,\lambda,\mu'\nu'}\,h_\lambda\,a_{\mu'\nu'}^{(3)}+\frac{v_0}{3}\,\square_{\mu\nu,\lambda,\mu'\nu'}\,\frac{\partial b_{\mu'\nu'}^{(1)}}{\partial x_\lambda}$$
$$+\sum_{k=1}^{3}\omega_{+2}^{(3k)}\,a_{\mu\nu}^{(k)}=0.$$

f) *2nd rank pseudo tensors* (11.6)

$$\frac{\partial b_{\mu\nu}^{(1)}}{\partial t}-\frac{\omega_{\mathrm{H}}}{\sqrt{2}}\,\overline{h_\mu\,a_\nu^{(2)}}+\frac{v_0}{3}\,\overline{\frac{\partial}{\partial x_\nu}\left(\sqrt{3}\,b_\mu^{(1)}+\sqrt{2}\,b_\mu^{(2)}+\frac{1}{\sqrt{10}}\,b_\mu^{(3)}\right)}-\frac{v_0}{3}\,\square_{\mu\nu,\lambda,\mu'\nu'}\,\frac{\partial a_{\mu'\nu'}^{(3)}}{\partial x_\lambda}+\omega_H\,\square_{\mu\nu,\lambda,\mu'\nu'}\,h_\lambda\,b_{\mu'\nu'}^{(1)}$$
$$+\sum_{k=1}^{2}\omega_{-2}^{(1k)}\,b_{\mu\nu}^{(k)}=0,$$

$$\frac{\partial b_{\mu\nu}^{(2)}}{\partial t}+\frac{3}{\sqrt{10}}\,\omega_H\,\overline{h_\mu\,a_\nu^{(4)}}-\frac{v_0}{3}\sqrt{2}\,\square_{\mu\nu,\lambda,\mu'\nu'}\,\frac{\partial a_{\mu'\nu'}^{(2)}}{\partial x_\lambda}+\frac{5}{3}\,\omega_H\,\square_{\mu\nu,\lambda,\mu'\nu'}\,h_\lambda\,b_{\mu'\nu'}^{(2)}\qquad+\sum_{k=1}^{2}\omega_{-2}^{(2k)}\,b_{\mu\nu}^{(k)}=0.$$

The equations for momentum (vector $\boldsymbol{a}^{(1)}$) and spin (pseudo vector $\boldsymbol{b}^{(1)}$) do not formulate the exact conservation of total angular momentum and the orientation of spin by (local) rotations of the gas. This point will be treated further in the following section.

§ 12. Discussion of conservation equations

For brevity we shall discuss the conservation equations in the absence of a magnetic field. Firstly we deal with the conservation of number of particles and energy. Using the physical interpretation of the expansion coefficients given in § 8, one obtains the familiar form of these conservation equations from the equation (11.1) for $a^{(1)}$ and $a^{(2)}$,
continuity equation:

$$\frac{\partial n}{\partial t} + \operatorname{div} n \langle \boldsymbol{v} \rangle = 0\,, \tag{12.1}$$

conservation of energy:

$$\frac{\partial}{\partial t}\left(\tfrac{3}{2}\, n\, k\, T\right) + \operatorname{div}\left(\boldsymbol{q} + \tfrac{5}{2}\, n k\, T \langle \boldsymbol{v} \rangle\right) = 0\,. \tag{12.2}$$

Next, we have a short look at the equation for the first pseudo scalar $b^{(1)}$ (11.2), although this is not a conservation equation. Using (8.8 and 10) this equation reads:

$$\frac{\partial}{\partial t} n \langle \hat{\boldsymbol{v}} \cdot \boldsymbol{s} \rangle + \operatorname{div} n \langle \boldsymbol{v}\, \hat{\boldsymbol{v}} \cdot \boldsymbol{s} \rangle + \omega_{+1}^{(11)}\, n \langle \hat{\boldsymbol{v}} \cdot \boldsymbol{s} \rangle = 0 \tag{12.3}$$

with $\hat{\boldsymbol{v}} = \boldsymbol{v}/v_0$. Thus $\omega_{+1}^{(11)}$ is the relaxation constant of the average longitudinal polarization (helicity) $\langle \hat{\boldsymbol{v}} \cdot \boldsymbol{s} \rangle$.

The conservation equations for momentum and angular momentum have to be discussed in more detail. The collision operator of the generalized BOLTZMANN equation (1.14) is local. Thus we cannot exactly derive from it the conservation equation for total angular momentum and no antisymmetric part of the pressure tensor shows up in the conservation equation for momentum. But we do have a relaxation of the spin in our theory, since the spin alone is not conserved. There is another effect which we cannot describe with a local collision operator (even in thermal equilibrium): the orientation of the spin by a local or uniform rotation of the system (BARNETT effect).

Ignoring the exact non-local collision operator, we directly amend our transport relaxation equations in two steps. Firstly, we add just one term in the equation of motion for $\langle \boldsymbol{s} \rangle$ which yields the correct BARNETT effect in local thermal equilibrium. This equation then already contains the correct antisymmetric part of the pressure tensor. Secondly, we add this antisymmetric part to the pressure tensor in the conservation equation for momentum. All the other equations will remain unchanged.

In order to find these additional terms, we proceed as follows. The distribution function $f_{0\omega}$ of a gas in equilibrium ($\beta_0 = 1/k\, T_0$) rotating with angular velocity $\boldsymbol{\omega}$ as a rigid body is proportional to $\exp - \beta_0 [\tfrac{1}{2}\, m\, v^2 - (\boldsymbol{l} + \hbar\, \boldsymbol{s})\,.\, \boldsymbol{\omega}]$, where $\boldsymbol{v}$ is the particle velocity in the lab-system, $\boldsymbol{l} = \boldsymbol{x} \times \boldsymbol{p}$ the orbital and $\hbar\, \boldsymbol{s}$ the intrinsic angular momentum of a particle. With the abbreviation $\boldsymbol{v}_\omega = \boldsymbol{\omega} \times \boldsymbol{x}$ ($\boldsymbol{x} = 0$ is a point on the axis of rotation), this distribution function reads:

$$f_{0\omega} = \left(\frac{m\, \beta_0}{2\, \pi}\right)^{\frac{3}{2}} n_0(x) \exp\left[-\tfrac{1}{2}\, \beta\, m (\boldsymbol{v} - \boldsymbol{v}_\omega)^2\right] \cdot \frac{\exp\{\beta_0\, \hbar\, \boldsymbol{\omega} \cdot \boldsymbol{s}\}}{\operatorname{tr} \exp\{\beta_0\, \hbar\, \boldsymbol{\omega} \cdot \boldsymbol{s}\}} \tag{12.4}$$

with

$$n_0(x) = N \exp\left[\tfrac{1}{2}\, \beta_0\, m\, v_\omega(x)^2\right] \Big/ \int \mathrm{d}^3x' \exp\left[\tfrac{1}{2}\, \beta_0 m\, v_\omega(x')^2\right].$$

N is the total number of particles. We assume ω to be small, then it is sufficient to retain terms linear in $\boldsymbol{\omega}$. Thus

$$\frac{\exp\{\beta_0\, \hbar\, \boldsymbol{\omega} \cdot \boldsymbol{s}\}}{\operatorname{tr} \exp\{\beta_0\, \hbar\, \boldsymbol{\omega} \cdot \boldsymbol{s}\}} \approx \frac{1}{2\, S + 1}\, (1 + \beta_0\, \hbar\, \boldsymbol{\omega} \cdot \boldsymbol{s})\,. \tag{12.5}$$

In this approximation we find for the BARNETT effect (cf. appendix A. 5)

$$\langle \boldsymbol{s} \rangle_{0\omega} = \tfrac{1}{3}\, S_0{}^2\, \beta_0\, \hbar\, \boldsymbol{\omega}; \quad S_0{}^2 = S(S+1)\,.$$

Expressing $\boldsymbol{\omega}$ by the rotation of the velocity field, $\operatorname{rot} \langle \boldsymbol{v} \rangle_{0\omega} = 2\, \boldsymbol{\omega}$, one also has

$$\langle \boldsymbol{s} \rangle_{0\omega} = \tfrac{1}{6}\, S_0{}^2\, \beta_0\, \hbar\, \operatorname{rot} \langle \boldsymbol{v} \rangle_{0\omega}\,. \tag{12.6}$$

Now we assume the same relation between $\langle \boldsymbol{s} \rangle$ and $\operatorname{rot} \langle \boldsymbol{v} \rangle$ to be valid in local equilibrium if the local $\operatorname{rot} \langle \boldsymbol{v} \rangle$ is small enough. Note, that in this linear approximation there is no alignment of $\langle \overline{s_\mu\, s_\nu} \rangle$ or higher spin tensors, since these terms would at least be quadratic in $\boldsymbol{\omega}$ or $\operatorname{rot} \langle \boldsymbol{v} \rangle$. Hence, in this approximation the transport equations for $\boldsymbol{a}^{(1)} \propto \langle \boldsymbol{v} \rangle$ and $\boldsymbol{b}^{(1)} \propto \langle \boldsymbol{s} \rangle$ only have to be amended.

We begin with the amended transport equation for the pseudovector $\boldsymbol{b}^{(1)}$. In the first equation of (11.4) we simply supplement the relaxation term by the following underlined expression which – according to the above remarks – is chosen in such a way as to give the correct Barnett effect in equilibrium:

$$\frac{\partial b_{\mu}^{(1)}}{\partial t} + \frac{v_0}{3} \frac{\partial b^{(1)}}{\partial x_{\mu}} - \frac{v_0}{\sqrt{6}} \varepsilon_{\mu\nu\lambda} \frac{\partial a_{\lambda}^{(2)}}{\partial x_{\nu}} + \frac{v_0}{\sqrt{3}} \frac{\partial b_{\mu\nu}^{(1)}}{\partial x_{\nu}}$$
$$+ \omega_{+1}^{(11)} \Big(b_{\mu}^{(1)} - \tfrac{1}{2} \underline{l_0\, \varepsilon_{\mu\nu\lambda} \frac{\partial a_{\lambda}^{(1)}}{\partial x_{\nu}}} \Big) + \sum_{k=2}^{3} \omega_{+1}^{(1k)} b_{\mu}^{(k)} = 0 . \tag{12.7}$$

The length l_0, according to (12.6) and (8.9 and 10), is given by

$$l_0 = \tfrac{1}{3} S_0 v_0 \beta_0 \hbar = \frac{S_0 \hbar}{m v_0} . \tag{12.8}$$

Thus, l_0 is proportional to the mean thermal de Broglie wave length of the particles. Obviously it is very small in practical cases.

Equation (12.7) may be rewritten in the more physical form, omitting the sum,

$$n_0 \frac{\partial \hbar \langle s_{\mu} \rangle}{\partial t} + \frac{\partial S_{\mu\nu}}{\partial x_{\nu}}$$
$$+ n_0\, \omega_{+1}^{(11)} \Big(\hbar \langle s_{\mu} \rangle - \tfrac{1}{2} m l_0^2 \varepsilon_{\mu\nu\lambda} \frac{\partial \langle v_{\lambda} \rangle}{\partial x_{\nu}} \Big) = 0 . \tag{12.9}$$

Herein, the spin flux tensor $S_{\mu\nu}$ is given by

$$S_{\mu\nu} = \frac{n_0}{\sqrt{3}} \hbar S_0 v_0 \Big(\tfrac{1}{3} b^{(1)} \delta_{\mu\nu} - \frac{1}{\sqrt{6}} \varepsilon_{\mu\nu\lambda} a_{\lambda}^{(2)} + \frac{1}{\sqrt{3}} b_{\mu\nu}^{(1)} \Big)$$
$$= n_0 \hbar \big(\tfrac{1}{3} \langle \boldsymbol{v} \cdot \boldsymbol{s} \rangle \delta_{\mu\nu} - \tfrac{1}{2} \langle v_{\mu} s_{\nu} - v_{\nu} s_{\mu} \rangle + \langle \overline{v_{\mu} v_{\nu}} \rangle$$
$$= n_0 \hbar \langle s_{\mu} v_{\nu} \rangle . \tag{12.10}$$

Next, we shall find the antisymmetric part of the pressure tensor. From the phenomenological theory (s. [11], p. 301, eq. 2.29) one knows that, due to conservation of total angular momentum, the relation holds:

$$\varepsilon_{\mu\nu\lambda} \Big(n_0 \frac{\partial \hbar \langle s_{\lambda} \rangle}{\partial t} + \frac{\partial S_{\lambda\varrho}}{\partial x_{\varrho}} \Big) = 2\, \widetilde{p_{\mu\nu}} = p_{\mu\nu} - p_{\nu\mu} . \tag{12.11}$$

Here, $\widetilde{p_{\mu\nu}}$ is the antisymmetric part of the pressure tensor. By comparison with (12.9) we see that

$$\widetilde{p_{\mu\nu}} = - \tfrac{1}{2} n_0\, \omega_{+1}^{(11)} \Big(\varepsilon_{\mu\nu\lambda} \hbar \langle s_{\lambda} \rangle - m l_0^2 \widetilde{\frac{\partial \langle v_{\nu} \rangle}{\partial x_{\mu}}} \Big),$$
$$\text{where} \quad \widetilde{\frac{\partial \langle v_{\nu} \rangle}{\partial x_{\mu}}} = \tfrac{1}{2} \Big(\frac{\partial \langle v_{\nu} \rangle}{\partial x_{\mu}} - \frac{\partial \langle v_{\mu} \rangle}{\partial x_{\nu}} \Big). \tag{12.12}$$

Note that $\widetilde{p_{\mu\nu}}$ vanishes as soon as the polarization equals the local Barnett polarization.

Now, we discuss the momentum equation. The first equation (11.3) again is a little incomplete. The full momentum equation must have the form [13]

$$\frac{\partial n m \langle v_{\mu} \rangle}{\partial t} + \frac{\partial p_{\mu\nu}}{\partial x_{\nu}} = 0 , \tag{12.13}$$

where $p_{\mu\nu}$ is the complete pressure tensor. The latter may be decomposed into its irreducible parts [14]:

$$p_{\mu\nu} = n k T \delta_{\mu\nu} + \widetilde{p_{\mu\nu}} + \overline{p_{\mu\nu}} . \tag{12.14}$$

The symmetric part $\overline{p_{\mu\nu}}$ (shear friction tensor) is given by

$$\overline{p_{\mu\nu}} = \frac{\sqrt{2}}{3} v_0^2 m n_0 a_{\mu\nu}^{(1)} = n m \langle \overline{v_{\mu} v_{\nu}} \rangle . \tag{12.15}$$

It is already contained in eq. (11.3). For the antisymmetric part $\widetilde{p_{\mu\nu}}$, which did not appear in (11.3) due to the locality of the collision operator, we have to take the expression (12.12), in order to ensure angular momentum conservation. We also may write $\widetilde{p_{\mu\nu}}$ in the following form

$$\widetilde{p_{\mu\nu}} = \eta_{\mathrm{r}} \Big(\widetilde{\frac{\partial \langle v_{\nu} \rangle}{\partial x_{\mu}}} - \frac{\hbar}{m l_0^2} \varepsilon_{\mu\nu\lambda} \langle s_{\lambda} \rangle \Big) . \tag{12.16}$$

The rotational viscosity η_{r} is proportional to the relaxation constant of the spin and given by

$$\eta_{\mathrm{r}} = \tfrac{1}{2} n_0\, \omega_{+1}^{(11)} m l_0^2 . \tag{12.17}$$

Finally we estimate the rotational viscosity by order of magnitude. Replacing $\omega_{+1}^{(11)}$ by an effective cross section σ_{r} according to

$$\omega_{+1}^{(11)} = n_0 v_0 \sigma_{\mathrm{r}} \tag{12.18}$$

one has

$$\eta_{\mathrm{r}} \approx n_0^2 l_0^2 \sigma_{\mathrm{r}} m v_0 . \tag{12.19}$$

On the other side, the shear viscosity η is related to an effective cross section σ by

$$\eta \approx \frac{1}{\sigma} m v_0 . \tag{12.20}$$

The ratio therefore is

$$\eta_{\mathrm{r}} / \eta \approx [n_0 l_0 \sqrt{\sigma \sigma_{\mathrm{r}}}]^2 . \tag{12.21}$$

As l_0 in practice is even smaller than the atomic diameter, the number in the bracket denotes the number of particles in a volume smaller than an atomic volume. This number is very small and the influence of the rotational viscosity in the momentum equation is entirely negligible as had, so to say, been anticipated by equation (11.3). –

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[13] In our linearized theory no convective momentum flux appears with a gas which is at rest in zeroth order.

[14] There is no bulk viscosity for dilute gases of particles with spin.

The shear viscosity η and the heat conductivity λ may be inferred from (11.3) and (11.5) as functions of certain relaxation coefficients $\omega_{PL}^{(ik)}$. This, however – including the effect of an external magnetic field on the heat conductivity and the viscosity: Senftleben effect – will be discussed in detail in a subsequent paper.

Appendix: Some spin traces

For the Cartesian components of the spin vector $\boldsymbol{s}$ holds

$$s_\mu s_\nu - s_\nu s_\mu = i\, \varepsilon_{\mu\nu\lambda}\, s_\lambda \tag{A.1}$$

$$s_\mu s_\mu = S(S+1) \equiv S_0^2, \tag{A.2}$$

where S is the magnitude of the spin.

From (A.1) and (A.2) and from invariance of the trace under rotation of the coordinate system follows:

$$\operatorname{tr} 1 = 2S+1 \tag{A.3}$$

$$\operatorname{tr} s_\mu = 0 \tag{A.4}$$

$$\operatorname{tr} s_\mu s_\nu = \frac{S_0^2}{3}\,(2S+1)\,\delta_{\mu\nu} \tag{A.5}$$

$$\operatorname{tr} s_\mu s_\nu s_\lambda = \frac{i}{6}\, S_0^2 (2S+1)\, \varepsilon_{\mu\nu\lambda} \tag{A.6}$$

$$\operatorname{tr} s_\mu s_\nu s_{\mu'} s_{\nu'} = \frac{S_0^2}{15}\,(2S+1) \cdot \left[(S_0^2 + \tfrac{1}{2})(\delta_{\mu\nu}\,\delta_{\mu'\nu'} + \delta_{\mu\nu'}\,\delta_{\nu\mu'}) + (S_0^2 - 2)\,\delta_{\mu\mu'}\,\delta_{\nu\nu'} \right]. \tag{A.7}$$

For

$$\overline{s_\mu s_\nu} = \tfrac{1}{2}(s_\mu s_\nu + s_\nu s_\mu) - \tfrac{1}{3} S_0^2\, \delta_{\mu\nu}$$

one finds the traces

$$\operatorname{tr} \overline{s_\mu s_\nu} = 0, \tag{A.8}$$

$$\operatorname{tr} \overline{s_\mu s_\nu s_\lambda} = 0, \tag{A.9}$$

$$\operatorname{tr} \overline{s_\mu s_\nu}\, \overline{s_{\mu'} s_{\nu'}} = \tfrac{2}{15}\, S_0^2\, S_1^2 (2S+1)\, \Delta_{\mu\nu,\mu'\nu'} \tag{A.10}$$

with

$$S_1^2 = S(S+1) - \tfrac{3}{4}.$$

Note that $S_1^2 = 0$ for spin $S = \frac{1}{2}$. For the definition of $\Delta_{\mu\nu,\mu'\nu'}$ see (7.8). Finally we need in the text

$$\begin{aligned} \operatorname{tr} \overline{s_\mu s_\nu}\, s_\lambda\, \overline{s_{\mu'} s_{\nu'}} &= -\operatorname{tr} s_\lambda\, \overline{s_\mu s_\nu}\, \overline{s_{\mu'} s_{\nu'}} \\ &= \tfrac{2}{15}\, i\, S_0^2\, S_1^2 (2S+1)\, \square_{\mu\nu,\lambda,\mu'\nu'}. \end{aligned} \tag{A.11}$$

For the definition of $\square_{\mu\nu,\lambda,\mu'\nu'}$ see (9.5).

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